The Proportional Ordinal Shapley Solution for Pure Exchange Economies*

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Abstract

We define the *proportional ordinal Shapley* (the *POSh*) solution, an ordinal concept for pure exchange economies in the spirit of the Shapley value. Our construction is inspired by Hart and Mas-Colell's (1989) characterization of the Shapley value with the aid of a potential function. The *POSh* exists and is unique and essentially single-valued for a fairly general class of economies. It satisfies individual rationality, anonymity, and properties similar to the null-player and null-player out properties in transferable utility games. Moreover, the *POSh* is immune to agents' manipulation of their initial endowments: It is not D-manipulable and does not suffer from the transfer paradox. Finally, we construct a bidding mechanism à la Pérez-Castrillo and Wettstein (2001) that implements the *POSh* in subgame perfect Nash equilibrium for economies where agents have homothetic preferences and positive endowments.

Keywords: Shapley Value; Exchange economy; Ordinal solution; Potential; Implementation

JEL Classification: D63; D50; C72

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1 Introduction

Economists have long been proposing allocation rules for economic environments and evaluating them by different desiderata. Though no rule is advantageous under every criterion, some allocation rules arise as dominant solution concepts for specific economic environments, such as the Walrasian allocation rule for pure exchange economies and the Shapley value (Shapley, 1953) for coalitional games with transferable utility (TU). A natural question is whether we can extend solution concepts that were initially designed for a specific economic environment to another.

In this paper, we propose a solution concept for pure exchange economies in the spirit of the Shapley value, which satisfies many appealing properties and is characterized by several methods in the class of TU games. Our construction is inspired by Hart and Mas-Colell's (1989) characterization of the Shapley value with the aid of a potential function. This function assigns a number to every TU game with the only condition that the marginal contributions to the potential of all players add up to the worth of the grand coalition. Hart and Mas-Colell (1989) prove the surprising fact that there is only one such potential function and the vector of marginal contributions coincides with the Shapley value.

We follow a similar approach and associate a number to each pure exchange economy, the *potential* of this economy. Due to the absence of a numeraire commodity in these environments, we choose each agent's initial endowment as a yardstick to measure the variation of his welfare in a solution. Moreover, to ensure the feasibility of the proposal, we assess an agent's marginal gain or loss in terms of the ratio of the potential of the economy over the potential of the subeconomy where he does not participate, instead of the difference between the two potentials. Then, the only condition that we impose to the potential function is the existence of an efficient allocation profile in the economy that satisfies that any agent is indifferent between that allocation and his "proportional" marginal contribution to the potential (that is, in terms of the ratio) times his initial endowment. That is, we require that it be possible for each agent to obtain his proportional marginal contribution to the potential through an efficient allocation.

The construction of the potential of a pure exchange economy entails the simultaneous definition of the efficient allocation profiles that are equivalent for all the agents to their proportional marginal contributions. These allocations are our solution for the economy. We name the set of these allocations the *proportional ordinal Shapley* (the *POSh*) solution. We include the word "ordinal" in the name of the solution because its first important characteristic is that, by construction, the *POSh* is an *ordinal solution*, that is, it is invariant to order-preserving transformations of the agents' utilities.

We show that the *POSh* solution is *unique* and *essentially single-valued*[1] in the set of exchange economies where the agents' preferences are reflexive, complete, transitive, strongly monotone, and continuous. It is also *individually rational.* Moreover, the *POSh* inherits several of the appealing properties of the Shapley value. In particular, it is *anonymous* with respect to the name of the agents (and it is also *neutral* with respect to the name of the commodities). Additionally, the *POSh* prescribes a zero bundle to any agent with zero endowments (these are "empty-bundle agents," we call them "empty agents" for short); that is, it satisfies the *empty-agent property.* Further, it satisfies the *empty-agent out property*, which requires that the presence of an empty agent does not influence the prescribed bundles for the rest of the agents. These properties are reminiscent of the null player property and the null player out property of the Shapley value (Derks and Haller, 1999).

Similar to the characterization of the Shapley value in terms of the Harsanyi's (1959) coalitional dividends, the *POSh* can be constructed and characterized using *coalitional dividend yield ratios.*

Additionally, we prove that the *POSh* is immune to certain peculiarities suffered by several allocation rules for pure exchange economies, such as the Walrasian equilibrium. First, the *POSh* is *not D-manipulable* (Postlewaite, 1979); that is, an agent cannot be better off by getting rid of part of his endowment. Second, it *does not suffer from the transfer paradox* (Postlewaite and Webb, 1984); that is, the transfer

[1] That is, if the *POSh* solution prescribes several allocations to an economy, every agent is indifferent among all these allocations. Moreover, any allocation that is indifferent for every agent to an allocation in the *POSh* solution is also in this set.

of a portion of his endowment to another individual cannot make an agent better off and the recipient worse off.

Finally, we provide an additional link between the *POSh* for pure exchange economies and the Shapley value for TU games in terms of their non-cooperative foundations. Pérez-Castrillo and Wettstein (2001) propose a bidding mechanism that implements the Shapley value. We adapt their mechanism[2] to our environment and show that it implements the *POSh* in subgame perfect Nash equilibrium (SPNE) for economies with an arbitrary number of agents in environments where the agents' preferences are homothetic.

The closest contribution to ours is the paper by Pérez-Castrillo and Wettstein (2006). They also provide an ordinal solution in the spirit of the Shapley value for pure exchange economies by extending the idea of Pazner and Schmeidler (1978), who introduce the notion of Pareto-efficient egalitarian equivalent (PEEE) allocations. A PEEE allocation is Pareto efficient and "fair" because, for each agent, it is equivalent preference-wise to the same fixed bundle. Pérez-Castrillo and Wettstein's (2006) ordinal Shapley value (*OSV*) considers possibly different individual endowments and is constructed so that it satisfies "consistency," in the sense that an agent's payoff is based on what he would obtain according to this value when applied to subeconomies.

An essential difference between the *POSh* and the *OSV* is in the domain of the solutions. We consider economies where the consumption bundles are non-negative, whereas the *OSV* is defined in environments where the consumption of a commodity can be positive or negative. Our set-up is more common in the general equilibrium literature and prevents the consumption of a negative amount of goods, such as apples. Let us note that most of the properties of the *POSh*, such as unicity, essential single-valueness, empty-agent, and empty-agent out, are not satisfied by the *OSV*. In addition, the *OSV* may suffer from the transfer paradox.

In addition to Pérez-Castrillo and Wettstein (2006), the early works by Harsanyi (1959), Shapley (1969), and Maschler and Owen (1992) propose extensions of the Shapley value to non-transferable utility environments such as the pure exchange

[2] See also Pérez-Castrillo and Wettstein (2002).

economy that we study. The three proposals are defined in the utility space. They abstract from the physical environment that generates the utilities. However, as Roemer (1986, 1988) discusses, much information is lost when one moves from the economic environment to the utility space. Thus, on the one hand, these proposals are not ordinal since the solutions are not invariant to alternative representations of the agents' utilities. Moreover, Greenberg et al. (2002) make the observation that the von Neumann and Morgenstein stable sets, defined for the economic environment and the utility space, respectively, may not coincide, even though both are ordinal. On the other hand, as Alon and Lehrer (2019) point out, two very different economic environments, whose solution should be different, may lead to the same allocation of utilities and, hence, the same solution.

McLean and Postlewaite (1989) also extend a notion from the class of TU games to the set of pure exchange economies. They provide an ordinal nucleolus, a solution concept proposed by Schmeidler (1969) for TU games. Nicolò and Perea (2005) and Alon and Lehrer (2019) offer ordinal solutions for bargaining problems.

The remainder of the paper is organized as follows. Section 2 describes the economic environment. It also introduces our new solution concept–the proportional ordinal Shapley solution. Section 3 proves the existence and uniqueness of the *POSh*. Several properties of the *POSh* are also stated and proved. Section 4 presents the bidding mechanism that implements the *POSh*. Section 5 concludes the paper. All the proofs are in the Appendix.

2 The environment and the solution concept

We consider a *pure exchange economy*. The set of agents is $N \equiv \{1, \dots, n\}$, with generic agent i. The set of goods is $L \equiv \{1, \dots, l\}$, which is fixed throughout this paper.

Agent i is described by $(\mathbf{w}_i, \succeq_i)$, where $\mathbf{w}_i \equiv (w_{i1}, \dots, w_{il}) \in \mathbb{R}^L_+$ is his commodity bundle, and $\succeq_i$ is his preference relation defined over $\mathbb{R}^L_+$. We assume $\succeq_i$ is reflexive, complete, and transitive for each $i \in N$.[3] We also assume that it is

[3] Agent i's preference $\succeq_i$ is reflexive if $\mathbf{x} \succeq_i \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^L_+$; $\succeq_i$ is complete if either $\mathbf{x} \succeq_i \mathbf{y}$

strongly monotone and continuous. Preference $\succeq_i$ is *strongly monotone* if $\mathbf{x} \succ_i \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$ such that $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. Preference $\succeq_i$ is *continuous* if $\{\mathbf{y} \in \mathbb{R}^L_+ \mid \mathbf{y} \succeq_i \mathbf{x}\}$ and $\{\mathbf{y} \in \mathbb{R}^L_+ \mid \mathbf{y} \preceq_i \mathbf{x}\}$ are closed subsets of $\mathbb{R}^L_+$, for all $\mathbf{x} \in \mathbb{R}^L_+$.

A pure exchange economy is a triplet $(N, \mathbf{w}, \succeq)$, where the vector $\mathbf{w}$ is understood as an endowment profile $(\mathbf{w}_1, \dots, \mathbf{w}_n)$ and $\succeq$ is understood as a preference profile $(\succeq_1, \dots, \succeq_n)$. For a fixed set of agents N, the set of all exchange economies where the agents' preferences are reflexive, complete, transitive, strongly monotone, and continuous is denoted by $\mathcal{E}^N$. The set of all such exchange economies with a finite set of agents is denoted by $\mathcal{E}$.

Definition 1. *A* ***feasible allocation*** *for an exchange economy* $(N, \mathbf{w}, \succeq)$ *is a profile* $\mathbf{z} \equiv (\mathbf{z}_1, \dots, \mathbf{z}_n) \in \mathbb{R}^{N \times L}_+$ *such that* $\sum_{i \in N} \mathbf{z}_i \leq \sum_{i \in N} \mathbf{w}_i$.

We denote by $Z(N, \mathbf{w}, \succeq)$ the set of feasible allocations for the exchange economy $(N, \mathbf{w}, \succeq)$.

Two feasible allocations are comparable when all agents prefer one to the other in unison. Formally, for $\mathbf{z}, \mathbf{z}' \in Z(N, \mathbf{w}, \succeq)$, we write $\mathbf{z} \succeq \mathbf{z}'$ if $\mathbf{z}_i \succeq_i \mathbf{z}'_i$ for all $i \in N$. Similarly, $\mathbf{z} \sim \mathbf{z}'$ if $\mathbf{z}_i \sim_i \mathbf{z}'_i$ for all $i \in N$. Then, we can define an efficient allocation.

Definition 2. *A feasible allocation* $\mathbf{z} \in Z(N, \mathbf{w}, \succeq)$ *is* ***efficient*** *if there is no feasible allocation* $\mathbf{z}' \in Z(N, \mathbf{w}, \succeq)$ *such that* $\mathbf{z}' \succeq \mathbf{z}$ *and* $\mathbf{z}'_j \succ_j \mathbf{z}_j$ *for some* $j \in N$.

We denote by $E(N, \mathbf{w}, \succeq)$ the set of efficient allocations for the exchange economy $(N, \mathbf{w}, \succeq)$.

We now define a solution concept for pure exchange economies.

Definition 3. *A* ***solution*** *is a correspondence* $F : \mathcal{E} \rightsquigarrow \bigcup_N \mathbb{R}^{N \times L}_+$ *such that* $F(N, \mathbf{w}, \succeq) \subseteq Z(N, \mathbf{w}, \succeq)$ *for all* $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.

Thus, a solution F assigns a set of feasible allocations to each pure exchange economy. Given two solutions F and F', we write $F \subseteq F'$ if $F(N, \mathbf{w}, \succeq) \subseteq F'(N, \mathbf{w}, \succeq)$ for all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.

A solution F is *single-valued* if F is a function, that is, it prescribes a unique feasible allocation for every economy. A solution F is *essentially single-valued* if $\{\mathbf{y} \in$

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or $\mathbf{y} \succeq_i \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$; $\succeq_i$ is transitive if $\mathbf{x} \succeq_i \mathbf{y}$ and $\mathbf{y} \succeq_i \mathbf{z}$ imply $\mathbf{x} \succeq_i \mathbf{z}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^L_+$.

$Z(N, \mathbf{w}, \succeq) \mid \mathbf{y} \sim \mathbf{x}\} = F(N, \mathbf{w}, \succeq)$ for all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$ and all $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$. Thus, an essentially single-valued solution prescribes a $\sim$-equivalence class within the set of all efficient allocations. For an essentially single-valued solution F, we write $F_i(N, \mathbf{w}, \succeq) \succeq_i F_i(N, \mathbf{w}', \succeq)$ for $i \in N$ if player i prefers the profiles in $F_i(N, \mathbf{w}, \succeq)$ to the profiles in $F_i(N, \mathbf{w}', \succeq)$. We write $F(N, \mathbf{w}, \succeq) \succeq F(N, \mathbf{w}', \succeq)$ similarly.

Given that agents have initial private endowments, a reasonable solution should ensure that an agent has an incentive to participate instead of walking away with his endowment. The individual rationality of a solution captures this notion:

Definition 4. *A solution F satisfies **individual rationality** if $\mathbf{x} \succeq \mathbf{w}$ for all $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$ and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.*

Next, we formulate two properties that adapt the ideas of the null player property and the null player out property (Derks and Haller, 1999) to pure exchange economies. We identify a type of agents in pure exchange economies who play a similar role as the null players in coalitional games. They are empty-basket agents; we call them empty agents. An agent $i \in N$ is an *empty agent* in the economy $(N, \mathbf{w}, \succeq)$ if $\mathbf{w}_i = \mathbf{0}$. An economy consisting of empty agents only is called an *empty economy*.

The definition of the second property requires the following notation. Let $\mathbf{x} \in \mathbb{R}_+^{N \times L}$ be an allocation profile. Then, for $N' \subseteq N$, we denote by $\mathbf{x}|_{N'} \in \mathbb{R}_+^{N' \times L}$ the profile $\mathbf{x}$ restricted to N', that is, $(\mathbf{x}|_{N'})_i = \mathbf{x}_i$ for all $i \in N'$. The restrictions of the preference profile are denoted analogously.

Definition 5. *A solution F satisfies the **empty-agent property** if $\mathbf{x}_i = \mathbf{0}$ for each empty agent $i \in N$ in $(N, \mathbf{w}, \succeq)$, all $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$, and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.*

Definition 6. *A solution F satisfies the **empty-agent out property** if $\mathbf{x}|_{N \setminus \{i\}} \in F(N \setminus \{i\}, \mathbf{w}|_{N \setminus \{i\}}, \succeq|_{N \setminus \{i\}})$ for each empty agent $i \in N$ in $(N, \mathbf{w}, \succeq)$, for all $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$, and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.*

The empty-agent and the empty-agent out properties are normative properties. The first one requires that an empty agent be entitled to a zero bundle in any allocation of the solution. In contrast, the empty-agent out property requires that the presence of an empty agent should not influence the allocation of the solution to the rest of the agents. In general, the two properties are logically independent of each other. But, in the presence of efficiency, the empty-agent out property implies the empty-agent property.[4]

It is worth mentioning that Shafer's (1980) example demonstrates that neither the empty-agent property nor the empty-agent out property is satisfied by Shapley's (1969) NTU value.

We now turn to the properties of anonymity and neutrality. The first refers to the agents and the second to the commodities. Before defining the property of anonymity, we introduce the notation for bijections of agents and economies.

Consider a bijection $\pi : N \to N'$. For a feasible allocation $\mathbf{z} \in Z(N, \mathbf{w}, \succeq)$, we define the allocation $\pi\mathbf{z} \in Z(N', \mathbf{w}, \succeq)$ by $(\pi\mathbf{z})_{\pi(i)} \equiv \mathbf{z}_i$ for all $i \in N$. Similarly, for a preference profile $\succeq$ for N, we define the preference profile $\succeq^\pi$ for N' by $\succeq^\pi_{\pi(i)} = \succeq_i$ for all $i \in N$. Then, for each economy $(N, \mathbf{w}, \succeq)$ and each bijection π, we denote the bijection of the economy by $\pi(N, \mathbf{w}, \succeq) \equiv (\pi[N], \pi\mathbf{w}, \succeq^\pi)$. That is, the structure of economy $\pi(N, \mathbf{w}, \succeq)$ is identical to $(N, \mathbf{w}, \succeq)$, but the names of the agents are changed according to π.

A solution is anonymous if the allocations that it prescribes for an economy are not influenced by the name of the agents. Formally:

Definition 7. *A solution F is* ***anonymous*** *if $\pi\mathbf{x} \in F\pi(N, \mathbf{w}, \succeq)$ for each bijection $\pi : N \to N'$ and each $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$.*

The property of neutrality, which refers to the name of the commodities, can be defined analogously. For a bijection $\rho : L \to L'$ and a commodity bundle $\mathbf{x} \in \mathbb{R}^L_+$, we define the commodity bundle $\rho\mathbf{x} \in \mathbb{R}^{L'}_+$ by $(\rho x)_{\rho(h)} \equiv x_h$ for all $h \in L$. Also, for a preference profile $\succeq$ over $\mathbb{R}^L_+$, the preference profile $\succeq^\rho$ is defined over $\mathbb{R}^{L'}_+$ by $\rho\mathbf{x} \succeq^\rho_i \rho\mathbf{y}$ if $\mathbf{x} \succeq_i \mathbf{y}$, for all $i \in N$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$. Then, for each

[4] To see this implication, consider an efficient solution that satisfies the empty-agent out property but does not satisfy the empty-agent property. Then there exists $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$ such that $\mathbf{x}_i \neq \mathbf{0}$ for some empty agent i in $(N, \mathbf{w}, \succeq)$. By the empty-agent out property, $\mathbf{x}|_{N\setminus\{i\}} \in F(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$. Then we could construct a feasible profile $\mathbf{y} \in Z(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ where $\mathbf{y}_j \equiv \mathbf{x}_j + \frac{\mathbf{x}_i}{n-1}$, which would be strictly preferred by every $j \in N\setminus\{i\}$ by strong monotonicity.

economy $(N, \mathbf{w}, \succeq)$ and each bijection ρ, we denote the bijection of the economy by $\rho(N, \mathbf{w}, \succeq) \equiv (N, \rho\mathbf{w}, \succeq^\rho)$. Thus, the structure of economy $\rho(N, \mathbf{w}, \succeq)$ is identical to $(N, \mathbf{w}, \succeq)$, but the names of the commodities are changed according to ρ.

Definition 8. *A solution F is **neutral** if $\rho\mathbf{x} \in F\rho(N, \mathbf{w}, \succeq)$ for each bijection $\rho : L \to L'$ and each $\mathbf{x} \in F(N, \mathbf{w}, \succeq)$.*

The last two properties that we propose concern the possibility for an agent to "manipulate" the solution outcome via his endowment. Aumann and Peleg (1974) demonstrate that before the opening of trade, an agent may be better off by getting rid of part of his endowment. In light of this peculiarity, Postlewaite (1979) formulates the following property, which is not implied by efficiency and individual rationality:

Definition 9. *An essentially single-valued solution F is **D-manipulable** if there exist $\mathbf{w}, \mathbf{w}' \in \mathbb{R}_+^{N\times L}$ such that $\mathbf{w}_i \geq \mathbf{w}'_i$ for some $i \in N$, $\mathbf{w}_j = \mathbf{w}'_j$ for each $j \in N\backslash\{i\}$, and $F_i(N, \mathbf{w}, \succeq) \prec_i F_i(N, \mathbf{w}', \succeq)$.*

An anomaly closely related to D-manipulability is the transfer paradox: a transfer of a portion of his endowment makes the donor better off and the recipient worse off (see, e.g., Postlewaite and Webb, 1984). Definition 10 formally states this paradox.

Definition 10. *An essentially single-valued solution F exhibits the **transfer paradox** if there exist $\mathbf{w}, \mathbf{w}' \in \mathbb{R}_+^{N\times L}$ and two distinct agents $i, j \in N$ such that $\mathbf{w}_i \geq \mathbf{w}'_i$, $\mathbf{w}_i + \mathbf{w}_j = \mathbf{w}'_i + \mathbf{w}'_j$ and $\mathbf{w}_k = \mathbf{w}'_k$ for each $k \in N\backslash\{i, j\}$, $F_i(N, \mathbf{w}, \succeq) \prec_i F_i(N, \mathbf{w}', \succeq)$, and $F_j(N, \mathbf{w}, \succeq) \succ_i F_j(N, \mathbf{w}', \succeq)$.*

Now we present our solution concept: the *proportional ordinal Shapley solution* (*POSh*). We define the *POSh* in terms of agents' preferences directly. Thus, it is an ordinal solution.

To define the *POSh*, we first define a *potential* in our economic environment, by adapting the idea of the potential introduced by Hart and Mas-Colell (1989) in TU games. In this class of games, a potential is a function that associates to every n-person game a single number. Once we have such a potential function, we can associate to each agent i in the game (N, v) his marginal contribution to the potential, that is, the difference between the potential of (N, v) and the potential of the game $(N \setminus \{i\}, v\mid_{N\setminus\{i\}})$. Then, it is also reasonable to request that the sum of these agents' marginal contributions be efficient, in the sense that it must be equal to the worth of the grand coalition. Hart and Mas-Colell (1989) show that there exists only one such potential function, and the vector of its marginal contributions corresponds to the Shapley value.

In our set of exchange economies, a potential function also associates a single number to each economy. To assign a surplus (an allocation) to each agent i in the economy $(N, \mathbf{w}, \succeq)$ based on the potential, we need a yardstick. We choose agent i's initial endowment $\mathbf{w}_i$ as the reference to measure agent i's welfare. Moreover, to ensure that the proposal made to this agent is feasible, we measure his marginal contribution not in terms of the difference between the potentials of the economies with and without him, $(N, \mathbf{w}, \succeq)$ and $(N \setminus \{i\}, \mathbf{w}\mid_{N\setminus\{i\}}, \succeq\mid_{N\setminus\{i\}})$, (times his initial endowment) because this could lead to a negative quantity of some commodities. Instead, we measure an agent's marginal gain or loss in terms of the ratio of the potentials. Finally, we require that it should be possible to allocate to each agent a bundle equivalent for him to the bundle corresponding to his marginal contribution to the potential (in terms of ratio) and that this allocation is efficient.

Thus, in our exchange economy, we define a potential function as follows:

Definition 11. *A* ***potential function*** $P : \mathcal{E} \to \mathbb{R}_{++}$ *is defined inductively on the number of players* $|N|$*:*

1. $P(\varnothing) \equiv 1$*;*

2. *for* $(N, \mathbf{w}, \succeq) \in \mathcal{E}$*,* $P(N, \mathbf{w}, \succeq)$ *satisfies that there exists* $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ *such that* $\frac{P(N,\mathbf{w},\succeq)}{P(N\setminus\{i\},\mathbf{w}\mid_{N\setminus\{i\}},\succeq\mid_{N\setminus\{i\}})}\mathbf{w}_i \sim_i \mathbf{x}_i$ *for all* $i \in N$*.*[5]

The prescription of the $POSh$ is intertwined with our definition of a potential. An allocation is in the $POSh$ if it is efficient and each agent i is indifferent between his prescribed bundle and some multiple of his endowment, where the multiple is equal to the change of potential resulting from his entrance. Thus, we have the

[5] If $N = \{i\}$, we let $P(N \setminus \{i\}, \mathbf{w}\mid_{N\setminus\{i\}}, \succeq\mid_{N\setminus\{i\}}) \equiv P(\varnothing)$.

following definition of a proportional ordinal Shapley solution in terms of a potential P.

Definition 12. *Given a potential function P, a* ***proportional ordinal Shapley solution*** $POSh : \mathcal{E} \rightsquigarrow \bigcup_N \mathbb{R}_+^{N\times L}$ *is defined by* $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$ *if* $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ *and* $\frac{P(N,\mathbf{w},\succeq)}{P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}})}\mathbf{w}_i \sim_i \mathbf{x}_i$ *for all* $i \in N$.

As we will see in the next section, the $POSh$ is an appealing solution concept. It enjoys many properties that echo the properties of the Shapley value, such as anonymity, the empty-agent property, and the empty-agent out property. Moreover, it is also immune to well-known anomalies of the Walrasian equilibrium, such as the D-manipulability and the transfer paradox.

At last, the $POSh$ is often easy to compute owing to its neat definition in terms of the potential. For illustration, we compute the $POSh$ for a simple 3-agent economy in Example 1.

Example 1. *Consider the economy with* $L = \{1, 2\}$, $N = \{1, 2, 3\}$, $\mathbf{w}_1 = \mathbf{w}_2 = (4, 4)$, $\mathbf{w}_3 = (2, 2)$, $u_1(x_1, y_1) = 4x_1+y_1$, $u_2(x_2, y_2) = x_2+4y_2$, *and* $u_3(x_3, y_3) = x_3y_3$.

To compute the $POSh(N, \mathbf{w}, u)$*, we need to find the potential of each subeconomy. First, it is easy to see that* $P(\{i\}) = 1$ *for* $i = 1, 2, 3$.[6] *Second, for the subeconomy* $(\{1, 2\}, (\mathbf{w}_1, \mathbf{w}_2, (u_1, u_2))$*, an efficient allocation where both agents obtain an allocation equivalent to* $\frac{P(\{1,2\})}{1}(4, 4)$ *assigns the eight units of the first commodity to agent* 1 *and those of the second commodity to agent* 2*. Hence,* $P(\{1, 2\})(4, 4) \sim_1 (8, 0)$ *(and* $P(\{1, 2\})(4, 4) \sim_2 (0, 8)$*), which implies that* $P(\{1, 2\}) = \frac{8}{5}$.

Third, any interior efficient allocation in the subeconomy $(\{1, 3\}, (\mathbf{w}_1, \mathbf{w}_3, (u_1, u_3))$ *satisfies that* $y_3 = 4x_3$*. Therefore, we can conjecture that an efficient allocation in the POSh is* $((6-x_3, 6-4x_3), (x_3, 4x_3))$ *such that* $0 \le x_3 \le \frac{6}{4}$*. Then* $P(\{1, 3\})(4, 4) \sim_1 (6 - x_3, 6 - 4x_3)$ *and* $P(\{1, 3\})(2, 2) \sim_1 (x_3, 4x_3)$*, that is,* $20P(\{1, 3\}) = 30 - 8x_3$ *and* $4P(\{1, 3\})^2 = 4x_3^2$*. Hence,* $P(\{1, 3\}) = \frac{15}{14}$*. Similarly,* $P(\{2, 3\}) = \frac{15}{14}$ *too.*

Finally, consider the economy $(N, 2\mathbf{w}, u)$*. We can conjecture that a generic efficient allocation in the POSh must satisfy* $x_1 = x_2$ *and* $y_1 = y_2 = 0$*, that is, it must be*

[6] In this example, we write $P(\{i\})$ instead of $P(\{i\}, \mathbf{w}_i, u_i)$, and similarly for the other subeconomies, for simplicity.

$((x_1, 0), (0, x_1), (10-x_1, 10-x_1))$, *for* $x_1 \in [0, 10]$. *Then,* $\frac{P(N)}{P(\{2,3\})}(4, 4) \sim_1 (x_1, 0)$ *and* $\frac{P(N)}{P(\{1,2\})}(2, 2) \sim_3 (10-x_1, 10-x_1))$. *This system of equations leads to* $P(N) = \frac{840}{497}$ *and* $x_1 = \frac{560}{71}$. *Therefore, the unique bundle in the POSh is* $((\frac{560}{71}, 0), (0, \frac{560}{71}), (\frac{150}{71}, \frac{150}{71}))$.

Remark 1. *It is easy to see that the Walrasian equilibrium allocation and the core for Example 1 coincide, which is* $((8, 0), (0, 8), (2, 2))$ *(the Walrasian equilibrium price is* $(1, 1)$*). Therefore, the POSh may not be in the core.*

3 Existence and properties of the proportional ordinal Shapley solution

In this section, we establish the existence, uniqueness, and other properties of the proportional ordinal Shapley solution.

Before we state our results regarding the *POSh*, we first prove the existence and uniqueness of the potential function restricted to economies in which each agent is not empty. We use the auxiliary notion of "coalitional dividend yield ratio" ("dividend ratio," for short), which is a multiplicative version of Harsanyi's (1959) coalitional dividend. Parallel to Hart and Mas-Colell (1989), our proof is also based on a simple representation of the potential through the dividend yield ratios.

Denote by $\mathcal{E}'$ the set of all economies with only non-empty agents. We establish in Proposition 1 the existence and uniqueness of the potential function restricted to $\mathcal{E}'$.

Proposition 1. *There exists a unique potential function restricted to* $\mathcal{E}'$.

Proposition 1 states the existence and uniqueness of the potential function if we restrict attention to economies without empty players. We make two remarks concerning the hypotheses that we use in the proposition.

Remark 2. *We state Proposition 1 for economies where the agents' preferences satisfy strong monotonicity. We cannot replace this hypothesis by the weaker axiom of strict monotonicity. Recall that player i's preference over commodities* $\succeq_i$ *is* ***strictly monotone*** *if* $\mathbf{x} \succ_i \mathbf{y}$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$ *such that* $x_h > y_h$ *for all*

$h \in L$. To see that this weaker property is not enough, consider a two-agent economy $(\{1,2\}, \mathbf{w}, \succeq)$, where $\mathbf{w}_1 = (0,1)$, $\mathbf{w}_2 = (1,0)$, $\succeq_1$ is represented by $u_1(x_1, y_1) = x_1$, and $\succeq_2$ is represented by $u_2(x_2, y_2) = y_2$. Both agents' preferences satisfy strict monotonicity but they do not satisfy strong monotonicity. According to Definition 11, $P(\{1\}, \mathbf{w}_1, \succeq_1) = P(\{2\}, \mathbf{w}_2, \succeq_2) = 0$. Then the denominators of both $\frac{P(\{1,2\},\mathbf{w},\succeq)}{P(\{2\},\mathbf{w}_2,\succeq_2)}$ and $\frac{P(\{1,2\},\mathbf{w},\succeq)}{P(\{1\},\mathbf{w}_1,\succeq_1)}$ vanish. Therefore, we are unable to assign a number to $P(\{1,2\}, \mathbf{w}, \succeq)$. Hence, a potential function does not exist for this economy.

Remark 3. *The full strength of the property of the continuity of preferences is not necessary for Proposition 1 to hold. The proof only requires lower semi-continuity of the preferences, i.e., $\{\mathbf{y} \in \mathbb{R}^L_+ \mid \mathbf{y} \preceq_i \mathbf{x}\}$ is closed for all $\mathbf{x} \in \mathbb{R}^L_+$ and all $i \in N$.*

In the proof of Proposition 1, we construct a system of dividend ratios that allows describing the potential of any economy. Then, Definition 12 together with the construction in the proof of Proposition 1 lead to the following representation of the *POSh* solution restricted to $\mathcal{E}'$ in terms of dividend ratios:

Corollary 1. *There exists a unique essentially single-valued proportional ordinal Shapley solution restricted to $\mathcal{E}'$.*

Furthermore, for all $(N, \mathbf{w}, \succeq) \in \mathcal{E}'$, there exists a vector of dividend yield ratios $(d_S)_{S \in 2^N \setminus \{\varnothing\}}$ such that for all $N' \in 2^N \setminus \{\varnothing\}$, $\mathbf{x} \in POSh(N', \mathbf{w}|_{N'}, \succeq|_{N'})$ if and only if $\mathbf{x} \in E(N', \mathbf{w}|_{N'}, \succeq|_{N'})$ and $\mathbf{x}_i \sim_i (\prod_{\substack{T \ni i \\ T \subseteq N'}} (1 + d_T))\mathbf{w}_i$ for all $i \in N'$.

Remark 4. *Pérez-Castrillo and Wettstein (2006) propose another ordinal solution for exchange economies, the ordinal Shapley value (OSV). They also provide a characterization of the OSV in terms of dividends. However, there is an important difference between their characterization and ours. For the OSV, the dividends d_S and d'_S of the same coalition $S \subseteq N'$ for an economy $(N, \mathbf{w}, \succeq)$ and its subeconomy $(N', \mathbf{w}\,|_{N'}, \succeq|_{N'})$, respectively, may be different. By contrast, for the POSh, the dividend ratios of the same coalition of an economy and its subeconomy are the same. In this sense, our characterization is closer in spirit to Harsanyi's (1959) characterization of the Shapley value in the set of TU games.*

We now consider the economies including empty agents. We note that the uniqueness of the potential function cannot be extended to the set of economies including empty agents. In particular, for an empty economy $(N, \mathbf{w}, \succeq)$, the potential of each subeconomy $(S, \mathbf{w}|_S, \succeq|_S)$ for $S \in 2^N \setminus \{\varnothing\}$ can be assigned an arbitrary positive number $P(S, \mathbf{w}|_S, \succeq|_S)$.

Given that the potential function and the *POSh* exist for economies without empty agents, it is useful to consider, for each economy, the subeconomy that contains only the set of non-empty agents of the original economy. Formally, we define the *support of the economy* $(N, \mathbf{w}, \succeq)$ as the subeconomy where an agent $i \in N$ participates if and only if $\mathbf{w}_i \neq \mathbf{0}$. The support of the economy $(N, \mathbf{w}, \succeq)$ is denoted by *supp*$(N, \mathbf{w}, \succeq)$. Similarly, we denote by $0(N, \mathbf{w}, \succeq)$ the subeconomy of $(N, \mathbf{w}, \succeq)$ where only the empty agents participate. Thus, each economy $(N, \mathbf{w}, \succeq)$ can be decomposed into two disjoint subeconomies: *supp*$(N, \mathbf{w}, \succeq)$ and $0(N, \mathbf{w}, \succeq)$.

Using the notion of the support of an economy, we can propose an extension of the potential function to the unrestricted domain as follows: (a) the potential of an economy consisting of empty agents only is equal to 1, and (b) the potential of an economy with both empty agents and non-empty agents is equal to the potential of its support. Moreover, we will show that this potential is associated with the unique essentially single-valued *POSh* of any economy with empty and non-empty agents. We will state these results in Theorem 1.

To establish the uniqueness of the *POSh*, we will use the relationship between the *POSh* of any pure exchange economy and the *POSh* of the support of that economy. We will also use the properties on empty agents that every *POSh* satisfies and that are stated and proven in Proposition 2.

Proposition 2. *Any proportional ordinal Shapley solution in $\mathcal{E}$ satisfies the empty-agent property and the empty-agent out property.*

Proposition 2 highlights that any *POSh* solution treats the empty agents as if they would not participate in the economy. They do not obtain any surplus (since they do not contribute to it) and they do not influence the sharing of the surplus allocated to the rest of the agents. Thus, Proposition 2 indicates that an empty agent can be viewed as a placeholder under any *POSh*.

Given that every proportional ordinal Shapley solution satisfies the empty-agent property and the empty-agent out property, its prescription for agents in a general economy can distinguish between empty and non-empty agents. On the one hand, an empty agent is prescribed a zero bundle by the empty-agent property. On the other hand, a non-empty agent is prescribed a bundle equal to some bundle prescribed by the *POSh* for the support of this economy by the empty-agent out property. Hence, we deduce the uniqueness of the *POSh* for the unrestricted domain from the uniqueness of the *POSh* for the economies without any empty agents.

Theorem 1. *There exists a unique essentially single-valued proportional ordinal Shapley solution in* $\mathcal{E}$*.*

Furthermore, for all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$*, there exists a vector of dividend yield ratios* $(d_S)_{S \in 2^N \setminus \{\varnothing\}}$ *such that for all* $N' \in 2^N \setminus \{\varnothing\}$*,* $\mathbf{x} \in POSh(N', \mathbf{w}|_{N'}, \succeq|_{N'})$ *if and only if* $\mathbf{x} \in E(N', \mathbf{w}|_{N'}, \succeq|_{N'})$ *and* $\mathbf{x}_i \sim_i (\prod_{\substack{T \ni i \\ T \subseteq N'}} (1 + d_T))\mathbf{w}_i$ *for all* $i \in N'$*.*

From here onward, we will refer to *the* proportional ordinal Shapley solution since it is unique.

We recall that Theorem 1 establishes the existence and uniqueness of the proportional ordinal Shapley solution for pure exchange economies where preferences are (in addition to reflexive, complete, and transitive) continuous and strongly monotone. The requirements for the existence of the *POSh* are incomparable with those for Walrasian equilibrium. Indeed, the existence of Walrasian equilibrium requires each agent's preference to be continuous, convex, and non-satiated, and each agent's endowment strictly positive (see Border, 2017). On the one hand, strong monotonicity is a stronger assumption than non-satiation. On the other hand, neither convex preferences nor strictly positive endowment is needed for the existence of the *POSh*.

We have seen that the *POSh* exists, and it is unique and essentially single-valued. Moreover, it satisfies the empty-agent and the empty-agent out properties. The last part of the section provides four additional properties of the *POSh*. First, we show that the *POSh* is individually rational.

Proposition 3. *The proportional ordinal Shapley solution satisfies individual rationality in* $\mathcal{E}$*.*

Second, we show that the *POSh* satisfies the properties of anonymity and neutrality. That is, it is immune to changes in the names of the agents and commodities.

Proposition 4. *The proportional ordinal Shapley solution satisfies anonymity and neutrality in $\mathcal{E}$.*

The last two properties of the $POSh$ state that it is robust against agents' manipulation of their initial endowment. Proposition 5 shows that an agent never has an incentive to throw away any part of his initial endowment, that is, the $POSh$ is not D-manipulable. Finally, Proposition 6 states that an agent is never better-off by transferring part of his initial endowment to another agent. Thus, the $POSh$ does not exhibit the transfer paradox.

Proposition 5. *The proportional ordinal Shapley solution is not D-manipulable in $\mathcal{E}$.*

Proposition 6. *The proportional ordinal Shapley solution does not exhibit the transfer paradox in $\mathcal{E}$.*

4 A mechanism implementing the proportional ordinal Shapley solution

In this section, we propose a new version of the Pérez-Castrillo and Wettstein's (2001) and (2002) bidding mechanism to implement the proportional ordinal Shapley solution.

In the original bidding mechanism, the agents first bid to each other to try to become the proposer. The proposer must honor his bids, but he earns the right to propose the division of the surplus, which can be either accepted or rejected.[7]

Given the defining characteristics of the $POSh$, our mechanism differs from previous proposals in two aspects: (i) a bid is interpreted as a promise to transfer a fixed proportion of resources; and (ii) in case of a rejection of his allocation plan, the proposer's payments due to his bid are delivered at the very end of the mechanism.

We implement the $POSh$ for any number of agents. However, we impose three additional constraints or modifications to the set of economies that we consider in

[7] Pérez-Castrillo and Wettstein (2005) also use a variant of this mechanism to implement the OSV for economies with at most three agents.

this section: (a) The agents' preferences are homothetic;[8] (b) no empty agent is present in the economy; and (c) the common domain of each agent i's preference is extended from $\mathbb{R}^L_+$ to $\mathbb{R}^L$ by letting $\mathbf{x} \prec_i \mathbf{y}$ if $\mathbf{x} \in \mathbb{R}^L \setminus \mathbb{R}^L_+$ and $\mathbf{y} \in \mathbb{R}^L_+$ and $\mathbf{x} \sim_i \mathbf{y}$ if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L \setminus \mathbb{R}^L_+$ for all $i \in N$. We extend the domain because, out of equilibrium, there may exist a solvency issue if the rejected proposer does not have enough endowment to pay the bids at the end of the mechanism. We denote by $\mathcal{E}^H$ the subset of economies where preferences are reflexive, complete, transitive, strongly monotone, and continuous, and that satisfy conditions (a)-(c) above.

We propose the following *proportional bidding mechanism* for $\mathcal{E}^H$:

For $|N| = 1$, for the economy $(\{i\}, \mathbf{w}_i, \succeq_i)$, the only agent i receives his own initial endowment $\mathbf{w}_i$.

For $|N| \geq 2$, we hypothesize that the mechanism has been defined for each economy $(N', \mathbf{w}', \succeq')$ with $|N'| < |N|$. Then the mechanism applied for an economy $(N, \mathbf{w}, \succeq)$ proceeds as follows:

$t = 1$: Each agent $i \in N$ submits a bid $b^N_{ij} \in \mathbb{R}_{++}$ for each agent $j \in N$, with $\prod_{j \in N} b^N_{ij} = 1$.

$t = 2$: Let the cumulative bid for agent $i \in N$ be denoted by $B^N_i \equiv \prod_{j \in N} b^N_{ji}$. An agent $\alpha \in \operatorname{argmax}_{i \in N} B^N_i$ is selected as the proposer by a non-degenerate lottery.[9] Then the proposer α puts forth an allocation plan $\mathbf{x}^N \in \mathbb{R}^{(N \setminus \{\alpha\}) \times L}$ specifying a bundle $\mathbf{x}^N_i \in \mathbb{R}^L_+$ for each agent $i \in N \setminus \{\alpha\}$.

$t = 3$: Each agent $i \in N \setminus \{\alpha\}$ accepts or rejects α's plan sequentially. We distinguish between two cases:

Case I: Every agent accepts α's plan. Then the grand coalition N forms, and the plan is implemented. The final outcome is that each agent $i \in N \setminus \{\alpha\}$ receives $\mathbf{x}^N_i$ and the proposer α receives the residue $\sum_{j \in N} \mathbf{w}_j - \sum_{i \in N \setminus \{\alpha\}} \mathbf{x}^N_i$.

[8] Agent i's preference $\succeq_i$ is **homothetic** if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L_+$ and all $\alpha \in \mathbb{R}_+$, $\mathbf{x} \succeq_i \mathbf{y}$ if and only if $\alpha\mathbf{x} \succeq_i \alpha\mathbf{y}$.

[9] A non-degenerate lottery selects each agent from $\operatorname{argmax}_{i \in N} B^N_i$ with a strictly positive probability.

Case II: Some agent rejects α's plan. Then the proposer forms his own standalone coalition $\{\alpha\}$. Moreover, the mechanism is applied to the subeconomy $(N \setminus \{\alpha\}, \mathbf{w}|_{N\setminus\{\alpha\}}, \succeq|_{N\setminus\{\alpha\}})$. Let $\mathbf{y}_i \in \mathbb{R}^L$ be the bundle received by agent $i \in N\setminus\{\alpha\}$ from the mechanism played by $(N\setminus\{\alpha\}, \mathbf{w}|_{N\setminus\{\alpha\}}, \succeq|_{N\setminus\{\alpha\}})$. On top of that, the proposer α transfers a commodity bundle $\left(\frac{\sqrt[n]{B_\alpha^N}}{b_{i\alpha}^N} - 1\right)\mathbf{y}_i$ to each agent $i \in N \setminus \{\alpha\}$. Therefore, the final commodity bundle is $\left(\frac{\sqrt[n]{B_\alpha^N}}{b_{i\alpha}^N}\right)\mathbf{y}_i$ for each $i \in N \setminus \{\alpha\}$, and it is $\mathbf{w}_\alpha - \sum_{i\in N\setminus\{\alpha\}}\left(\frac{\sqrt[n]{B_\alpha^N}}{b_{i\alpha}^N} - 1\right)\mathbf{y}_i$ for the proposer α.

Before presenting the main result of this section, we provide a characterization of the proportional ordinal Shapley solution in terms of "proportional concessions," which bears some resemblance with the original Pérez-Castrillo and Wettstein's (2006) definition of the OSV. The characterization is interesting by itself. It will also allow us to simplify the proof of the implementation theorem.

Definition 13 proposes a solution ζ for $\mathcal{E}$, and Proposition 7 states that it coincides with the $POSh$.

Definition 13. *The solution $\zeta : \mathcal{E} \rightsquigarrow \bigcup_N \mathbb{R}_+^{N\times L}$ is defined recursively on the number of agents $|N|$ as follows:*

1. *For $|N| = 1$, i.e., $N = \{i\}$, $\zeta(\{i\}, \mathbf{w}_i, \succeq_i) \equiv \{\mathbf{w}_i\}$.*

2. *For $|N| \geq 2$, we hypothesize that ζ has been defined and is essentially single-valued for each economy $(N', \mathbf{w}', \succeq')$ with $|N'| < |N|$. Then, $\mathbf{x} \in \zeta(N, \mathbf{w}, \succeq)$ if $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ and there exists a concession vector $(c_{ij}^N)_{j\in N\setminus\{i\}} \in \mathbb{R}^{N\setminus\{i\}}$ for each $i \in N$ that satisfies:*

   (a) $\prod_{j\in N\setminus\{i\}} c_{ij}^N = \prod_{j\in N\setminus\{i\}} c_{ji}^N$ for each $i \in N$; and

   (b) for each $j \in N \setminus \{i\}$, there exists $a_{ij}^N \in \mathbb{R}$ such that $a_{ij}^N \mathbf{w}_j \sim_j \zeta_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ and $\mathbf{x}_j \sim_j c_{ij}^N a_{ij}^N \mathbf{w}_j$.

We can read part (2b) of Definition 13 as follows. Agent j must be indifferent between the bundle that the solution prescribes to him ($\mathbf{x}_j$) and a bundle ($a_{ij}^N \mathbf{w}_j$) that is equivalent to what he can obtain without agent i according to the solution, boosted by the concession c_{ij}^N of agent i to agent j. Condition (2a) states the "fairness" requirement that the concessions that an agent receives in total (which in our framework corresponds to their product) be the same as the concessions that he makes to the other agents.

Proposition 7 states the characterization of the *POSh* in terms of concessions.

Proposition 7. *The proportional ordinal Shapley solution coincides with the solution ζ.*

An implication of Proposition 7 is that the vector of concessions for an economy $(N, \mathbf{w}, \succeq)$ is unique, given that the *POSh* is essentially single-valued.

Theorem 2 uses Proposition 7 to show that the proportional ordinal bidding mechanism implements the *POSh* in subgame perfect Nash equilibrium in pure strategies (SPNE) when the agents' preferences are homothetic, and their endowments are not zero. In the proof, we relate the equilibrium bids in the mechanism and the concessions in Definition 13.

Theorem 2. *The proportional bidding mechanism implements the proportional ordinal Shapley solution in SPNE in the set of economies with homothetic preferences and without empty agents.*

5 Conclusion

We espouse a new ordinal solution concept for pure exchange economies, the *POSh* solution. Its construction is inspired by the potential function, which allows a nice characterization of the Shapley value in TU games. The *POSh* solution satisfies properties similar to the Shapley value, such as efficiency, anonymity, and properties related to null players. It is also individually rational and does not suffer from agents' manipulation of their initial endowment.

We further highlight the link between the *POSh* for pure exchange economies and the Shapley value for TU games through their implementation. We show that a variant of a mechanism that implements the Shapley value implements the *POSh* for the particular environments where agents' preferences are homothetic.

One natural avenue for future research is extending our solution concept and its properties to pure exchange economies with a continuum of agents of finite types. It is easy to extend the notions of the potential and the proportional ordinal Shapley solution to these economies. However, the analysis of the properties of the $POSh$ in these environments is outside the scope of this paper.

Appendix

Proof of Proposition 1. First, we show that there exists at most one potential function. Suppose otherwise, that is, suppose that there exist two distinct potential functions P and P'. Then, without loss of generality, assume that for $(N, \mathbf{w}, \succeq)$, it happens that $P(N, \mathbf{w}, \succeq) > P'(N, \mathbf{w}, \succeq)$ and $P(S, \mathbf{w}|_S, \succeq|_S) = P'(S, \mathbf{w}|_S, \succeq|_S)$ for all $S \in 2^N \setminus \{N\}$. This implies that there exist two allocations $\mathbf{x}, \mathbf{y} \in E(N, \mathbf{w}, \succeq)$ such that $\mathbf{x}_k \sim_k \frac{P(N,\mathbf{w},\succeq)}{P(N\setminus\{k\},\mathbf{w}|_{N\setminus\{k\}},\succeq|_{N\setminus\{k\}})}\mathbf{w}_k \succ_k \frac{P'(N,\mathbf{w},\succeq)}{P'(N\setminus\{i\},\mathbf{w}|_{N\setminus\{k\}},\succeq|_{N\setminus\{k\}})}\mathbf{w}_k \sim_k \mathbf{y}_k$ for all $k \in N$, where the strict preference follows from strong monotonicity and the premise on P and P'. However, this contradicts the premise that $\mathbf{y} \in E(N, \mathbf{w}, \succeq)$. Therefore, there exists at most one potential function.

Second, to prove the existence of a potential function, we construct inductively a system of dividend ratios $(d_S)_{S\in 2^N\setminus\{\varnothing\}}$ for each economy $(N, \mathbf{w}, \succeq) \in \mathcal{E}^N$:

1. For $|S| = 1$, $d_S \equiv 0$;

2. for $|S| \geq 2$, we hypothesize that d_T has been defined for each $T \in 2^S \setminus \{\varnothing\}$. Then, we define $d_S \equiv \sup\{d \in [-1, +\infty) \mid \exists \mathbf{x} \in Z(S, \mathbf{w}\,|_S, \succeq|_S)$ such that $(1 + d)(\prod_{\substack{T\ni i\\ T\subsetneq S}}(1 + d_T))\mathbf{w}_i \sim_i \mathbf{x}_i$ for all $i \in S\}$.

Notice that d_S is well-defined for $|S| \geq 2$. Indeed, we check that the supremum operates on a non-empty set: $d = -1$ is in the set since $(1-1)(\prod_{\substack{T\ni i\\ T\subsetneq S}}(1+d_T))\mathbf{w}_i \sim_i \mathbf{0}$ for all $i \in S$ and $\mathbf{0} \in Z(S, \mathbf{w}\,|_S, \succeq|_S)$.

Next, we claim that d_S satisfies that $(\prod_{\substack{T\ni i\\ T\subseteq S}}(1 + d_T))\mathbf{w}_i \sim_i \mathbf{x}_i$ for all $i \in S$ and some $\mathbf{x} \in E(S, \mathbf{w}\,|_S, \succeq|_S)$. Note that d_S satisfies that there exists $\mathbf{x} \in E(S, \mathbf{w}|_S, \succeq|_S)$ such that $(\prod_{\substack{T\ni i\\ T\subseteq S}}(1 + d_T))\mathbf{w}_i \preceq_i \mathbf{x}_i$ for all $i \in S$ because each agent's preference is continuous and $Z(S, \mathbf{w}\,|_S, \succeq|_S)$ is closed. We prove our claim by contradiction:

if there exists $k \in S$ such that $(\prod_{\substack{T \ni k \\ T \subseteq S}} (1 + d_T))\mathbf{w}_k \prec_k \mathbf{x}_k$, then it is possible to construct an alternative feasible allocation profile $\mathbf{y} \in Z(S, \mathbf{w} \mid_S, \succeq\mid_S)$ such that $(\prod_{\substack{T \ni i \\ T \subseteq S}} (1+d_T))\mathbf{w}_i \prec_i \mathbf{y}_i$ for all $i \in S$. The existence of the profile $\mathbf{y}$ would imply that the supremum was not attained at d_S since d_S could be increased by a sufficiently small amount without violating feasibility. To construct $\mathbf{y}$ from $\mathbf{x}$, first note that $\mathbf{0} \preceq_k (\prod_{\substack{T \ni k \\ T \subseteq S}} (1 + d_T))\mathbf{w}_k \prec_k \mathbf{x}_k$, hence $x_{kh} > 0$ for some $h \in L$. Define $\mathbf{y}$ by

$$y_{ig} \equiv \begin{cases} x_{ig} & \text{if } i \in S \text{ and } g \in L \setminus \{h\}, \\ x_{ig} - \epsilon & \text{if } i = k \text{ and } g = h, \\ x_{ig} + \frac{\epsilon}{|S|-1} & \text{if } i \in S \setminus \{k\} \text{ and } g = h, \end{cases}$$

where $\epsilon \in \mathbb{R}_{++}$ is sufficiently small so that $(\prod_{\substack{T \ni k \\ T \subseteq S}} (1 + d_T))\mathbf{w}_k \prec_k \mathbf{y}_k$ and $y_{kh} \geq 0$. By strong monotonicity, we have $(\prod_{\substack{T \ni i \\ T \subseteq S}} (1 + d_T))\mathbf{w}_i \prec_i \mathbf{y}_i$ for all $i \in S$.

Therefore, we have proven the existence of dividend ratios $(d_S)_{S \in 2^N \setminus \{\varnothing\}}$ that satisfy that, for each $S \in 2^N \setminus \{\varnothing\}$, there exists $\mathbf{x} \in E(S, \mathbf{w} \mid_S, \succeq\mid_S)$ such that $(\prod_{\substack{T \ni i \\ T \subseteq S}} (1 + d_T))\mathbf{w}_i \sim_i \mathbf{x}_i$ for all $i \in S$.

We can now construct the potential function:

$$P(N, \mathbf{w}, \succeq) \equiv \prod_{S \in 2^N \setminus \{\varnothing\}} (1 + d_S)$$

for $N \neq \varnothing$ and $P(\varnothing) = 1$. The function $P(N, \mathbf{w}, \succeq)$ satisfies the conditions in Definition 11 given the construction of the dividend ratios. This establishes the existence of a potential function restricted to $\mathcal{E}'$. □

Proof of Corollary 1. The existence, uniqueness, and essential single-valuedness of $POSh$ restricted to $\mathcal{E}'$ follows from Proposition 1 and Definition 12. The representation in terms of dividend ratios follows from $\frac{P(N', \mathbf{w}|_{N'}, \succeq|_{N'})}{P(N' \setminus \{i\}, \mathbf{w}|_{N' \setminus \{i\}}, \succeq|_{N' \setminus \{i\}})} = \frac{\prod_{T \subseteq N'} (1+d_T)}{\prod_{T \subseteq N' \setminus \{i\}} (1+d_T)} = \prod_{\substack{T \ni i \\ T \subseteq N'}} (1 + d_T)$. □

Proof of Proposition 2. The empty-agent property follows from the efficiency implied by Definition 12, once we will prove the empty-agent out property, which we now do.

First, we claim that any potential function satisfies

$$P(N, \mathbf{w}, \succeq) = P(supp(N, \mathbf{w}, \succeq))P(0(N, \mathbf{w}, \succeq)). \tag{1}$$

We prove equation (1) by induction on p, by which we denote the number of non-empty agents of an economy $(N, \mathbf{w}, \succeq)$ with q empty agents (q is an arbitrary fixed positive number). The equation holds trivially for an economy with only q empty agents, i.e, when $p = 0$. Now we consider an economy $(N, \mathbf{w}, \succeq)$ with $p \geq 1$ non-empty agents and q empty agents. Denote by $\mathbf{x} \in E(supp(N, \mathbf{w}, \succeq))$ an allocation profile satisfying that $\mathbf{x}_i \sim_i \frac{P(supp(N,\mathbf{w},\succeq))}{P(supp(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))}\mathbf{w}_i$ for all non-empty agent i. The allocation $\mathbf{x}$ satisfies that for each non-empty agent i,

$$
\begin{aligned}
\mathbf{x}_i \sim_i & \frac{P(supp(N,\mathbf{w},\succeq))P(0(N,\mathbf{w},\succeq))}{P(supp(N\setminus\{i\},\mathbf{w}\mid_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))P(0(N,\mathbf{w},\succeq))}\mathbf{w}_i \\
= & \frac{P(supp(N,\mathbf{w},\succeq))P(0(N,\mathbf{w},\succeq))}{P(supp(N\setminus\{i\},\mathbf{w}\mid_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))P(0(N\setminus\{i\},\mathbf{w}\mid_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))}\mathbf{w}_i \\
= & \frac{P(supp(N,\mathbf{w},\succeq))P(0(N,\mathbf{w},\succeq))}{P(N\setminus\{i\},\mathbf{w}\mid_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}})}\mathbf{w}_i,
\end{aligned}
$$

where the first equality follows from the premise that i is not an empty agent and the second from the induction hypothesis (there exist $p-1$ non-empty agents in $(N\setminus\{i\}, \mathbf{w}\mid_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$). Then consider a new allocation profile $\mathbf{y} \in E(N, \mathbf{w}, \succeq)$, where $\mathbf{y}_j = \mathbf{x}_j$ for each non-empty agent j and $\mathbf{y}_k = \mathbf{0}$ for each empty agent k. Notice that the constructed profile $\mathbf{y}$ satisfies that $\mathbf{y}_i \sim_i \frac{P^o(N,\mathbf{w},\succeq)}{P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}})}\mathbf{w}_i$ for all $i \in N$, where we define $P^o(N, \mathbf{w}, \succeq) = P(supp(N, \mathbf{w}, \succeq))P(0(N, \mathbf{w}, \succeq))$. Moreover, by strong monotonicity, $p \geq 1$, and an argument similar to that establishing the uniqueness of the potential function restricted to $\mathcal{E}'$ in Proposition 1, we have that the numerical value of the potential is unique, hence $P^o(N, \mathbf{w}, \succeq) = P(N, \mathbf{w}, \succeq)$. Finally, since q is arbitrary, we have proven the equation (1), which immediately implies the empty-agent out property of any $POSh$. □

Proof of Theorem 1. Let us denote $POSh'$ the proportional ordinal Shapley solution restricted to $\mathcal{E}'$, which is unique and essentially single-valued by Corollary 1. First, by Proposition 2, any $POSh$ for $\mathcal{E}$ satisfies the empty-agent property and the empty-agent out property. Therefore, for any $POSh$ and any $(N, \mathbf{w}, \succeq)$,

$$
POSh_i(N,\mathbf{w},\succeq) \equiv \begin{cases} \mathbf{0} & \text{if } \mathbf{w}_i = \mathbf{0}, \\ POSh'_i(supp(N,\mathbf{w},\succeq)) & \text{if } \mathbf{w}_i \neq \mathbf{0}. \end{cases} \tag{2}
$$

Hence, if $POSh$ exists for $\mathcal{E}$, it is also unique and essentially single-valued.

Second, let us denote P' the potential associated with $POSh'$ in $\mathcal{E}'$. We now propose the following potential function $P : \mathcal{E} \to \mathbb{R}$:

$$P(N, \mathbf{w}, \succeq) \equiv \begin{cases} 1 & \text{if } \mathbf{w}_i = \mathbf{0} \text{ for all } i \in N \\ P'(supp(N, \mathbf{w}, \succeq)) & \text{otherwise.} \end{cases} \tag{3}$$

We show that the function P can be associated with the $POSh$ that we constructed in (2) for $\mathcal{E}$; that is, $\frac{P(N,\mathbf{w},\succeq)}{P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))}\mathbf{w}_i \sim_i \mathbf{x}_i$ for all $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$ and all $i \in N$. If $\mathbf{w}_i = 0$, then the result is immediate because $POSh_i(N, \mathbf{w}, \succeq) = 0$. Otherwise, consider an economy $(N, \mathbf{w}, \succeq)$ where i is a non-empty agent, and $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$. Then, equation (2) states that $\mathbf{x} \in POSh'_i(supp(N, \mathbf{w}, \succeq))$. Therefore, $\mathbf{x}_i \sim_i \frac{P'(supp(N,\mathbf{w},\succeq))}{P'(supp(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))}\mathbf{w}_i = \frac{P(N,\mathbf{w},\succeq)}{P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))}\mathbf{w}_i$.

Moreover, let $N' \subseteq N$ be the set of non-empty agents in $(N, \mathbf{w}, \succeq)$. Then $\mathbf{x} \in E(N, \mathbf{w}, \succeq)$ and $\mathbf{x}_i \equiv \mathbf{0}$ for all $i \in N \setminus N'$ if and only if $\mathbf{x}|_{N'} \in E(supp(N, \mathbf{w}, \succeq))$. Thus, the constructed $P : \mathcal{E} \to \mathbb{R}$ is a potential function associated with the $POSh$ for $\mathcal{E}$, which means that there exists a $POSh$ for $\mathcal{E}$.

Finally, we show the existence of the vector of dividend ratios. For the coalitions without empty agents, that is, in the $supp(N, \mathbf{w}, \succeq)$), we take the vector found in Corollary 1. Additionally, we define $d_S \equiv 0$ for each $S \in 2^N \setminus \{\varnothing\}$ if there exists an empty agent in S.

To verify that the previous vector of dividend ratios satisfies the condition stated in the theorem, it suffices to show that $P(N, \mathbf{w}, \succeq) = \prod_{T\in 2^N\setminus\{\varnothing\}}(1+d_T)$ for a general economy $(N, \mathbf{w}, \succeq)$. We prove this by induction on the number of non-empty agents. It is easy to see that $P(N, \mathbf{w}, \succeq) = \prod_{T\in 2^N\setminus\{\varnothing\}}(1 + d_T) = 1$ holds when $(N, \mathbf{w}, \succeq)$ consists of empty agents only. Now consider an economy $(N, \mathbf{w}, \succeq)$ in which i is a non-empty agent. Let $N' \subseteq N$ be the set of all non-empty agents. Then, for any $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$, $\mathbf{x}_i \sim_i \frac{P(N,\mathbf{w},\succeq)}{P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))}\mathbf{w}_i = \frac{P'(supp(N,\mathbf{w},\succeq))}{P'(supp(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))}\mathbf{w}_i = \frac{\prod_{T\in 2^{N'}\setminus\{\varnothing\}}(1+d_T)}{\prod_{T\in 2^{N'\setminus\{i\}}\setminus\{\varnothing\}}(1+d_T)}\mathbf{w}_i = \frac{\prod_{T\in 2^{N}\setminus\{\varnothing\}}(1+d_T)}{\prod_{T\in 2^{N\setminus\{i\}}\setminus\{\varnothing\}}(1+d_T)}\mathbf{w}_i = \frac{\prod_{T\in 2^{N}\setminus\{\varnothing\}}(1+d_T)}{P(N\setminus\{i\},\mathbf{w}|_{N\setminus\{i\}},\succeq|_{N\setminus\{i\}}))}\mathbf{w}_i$, where the last equality follows from the induction. Thus, $P(N, \mathbf{w}, \succeq) = \prod_{T\in 2^N\setminus\{\varnothing\}}(1+d_T)$. □

Proof of Proposition 3. By Theorem 1, we have that $\mathbf{x}_i \sim_i (\prod_{\substack{T\ni i\\ T\subseteq N}}(1 + d_T))\mathbf{w}_i$ for all $i \in N$, all $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$, and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$, where $(d_S)_{S\in 2^N\setminus\{\varnothing\}}$ is the vector of dividend ratios corresponding to $POSh(N, \mathbf{w}, \succeq)$. We show that $\mathbf{x}_i \sim_i$

$(\prod_{\substack{T\ni i\\T\subseteq N}}(1+d_T))\mathbf{w}_i \succeq_i \mathbf{w}_i$ by induction on $|N|$. Since $POSh_i(\{i\},\mathbf{w}_i,\succeq_i)=\{\mathbf{w}_i\}$ for $N=\{i\}$, our assertion trivially holds for $|N|=1$.

For $|N|>1$, assume that the property holds for any economy with less than $|N|$ agents. Suppose now, by contradiction, that it does not hold for $(N,\mathbf{w},\succeq)$, that is, there exists $i\in N$ such that $\mathbf{x}_i \sim_i (\sum_{\substack{T\ni i\\T\subseteq N}}(1+d_T))\mathbf{w}_i \prec_i \mathbf{w}_i$. Then there must exist $j\in N\setminus\{i\}$ such that $\mathbf{x}_j \succ_j (\prod_{\substack{T\ni j\\T\subseteq N\setminus\{i\}}}(1+d_T))\mathbf{w}_j$. The existence of such an agent j follows from $\mathbf{x}\in E(N,\mathbf{w},\succeq)$, $\mathbf{x}_i \prec_i \mathbf{w}_i$, and the feasibility of the allocation that assigns agent i with $\mathbf{w}_i$ and the rest of agents with a bundle prescribed by $POSh(N\setminus\{i\},\mathbf{w}\mid_{N\setminus\{i\}},\succeq\mid_{N\setminus\{i\}})$, which is individually rational by the induction hypothesis. Therefore, there exists $j\in N\setminus\{i\}$ such that $(\prod_{\substack{T\ni j\\T\subseteq N}}(1+d_T))\mathbf{w}_j \sim_j \mathbf{x}_j \succ_j (\prod_{\substack{T\ni j\\T\subseteq N\setminus\{i\}}}(1+d_T))\mathbf{w}_j$. Agent j's strict preference $(\prod_{\substack{T\ni j\\T\subseteq N}}(1+d_T))\mathbf{w}_j \succ_j (\prod_{\substack{T\ni j\\T\subseteq N\setminus\{i\}}}(1+d_T))\mathbf{w}_j$ implies that $\prod_{\substack{T\ni i,j\\T\subseteq N}}(1+d_T)>1$ by strong monotonicity.

By the induction hypothesis, we have $(\prod_{\substack{T\ni i\\T\subseteq N\setminus\{j\}}}(1+d_T))\mathbf{w}_i \succeq_i \mathbf{w}_i$. Together with the inequality $\prod_{\substack{T\ni i,j\\T\subseteq N}}(1+d_T)>1$, it implies that $\mathbf{x}_i \sim_i (\prod_{\substack{T\ni i\\T\subseteq N}}(1+d_T))\mathbf{w}_i = (\prod_{\substack{Q\ni i,j\\Q\subseteq N}}(1+d_Q))(\prod_{\substack{T\ni i\\T\subseteq N\setminus\{j\}}}(1+d_T))\mathbf{w}_i \succeq_i (\prod_{\substack{T\ni i\\T\subseteq N\setminus\{j\}}}(1+d_T))\mathbf{w}_i \succeq_i \mathbf{w}_i$, which contradicts our assumption. Therefore, the $POSh$ satisfies individual rationality. $\square$

Proof of Proposition 4. We prove anonymity and neutrality separately. We first prove anonymity of the $POSh$. It is easy to see that the efficient allocation correspondence is anonymous, that is, $\mathbf{x}\in E(N,\mathbf{w},\succeq)$ if and only if $\pi\mathbf{x}\in E\pi(N,\mathbf{w},\succeq)$, for all bijection $\pi: N\to N'$ and all $(N,\mathbf{w},\succeq)\in\mathcal{E}$.

Consider the economies $(N,\mathbf{w},\succeq)$ and $\pi(N,\mathbf{w},\succeq)$. Take $\mathbf{x}\in POSh(N,\mathbf{w},\succeq)$ and let $(d_T)_{T\in 2^N\setminus\{\varnothing\}}$ be its vector of dividend ratios. We are going to show that $\pi\mathbf{x}\in POSh\pi(N,\mathbf{w},\succeq)$ by proving that $(d'_{\pi[T]})_{T\in 2^N\setminus\{\varnothing\}}$, with $d'_{\pi[T]}=d_T$ for all $T\in 2^N\setminus\{\varnothing\}$, constitutes a vector of dividend ratios for $\pi\mathbf{x}$. Indeed, for each $S\in 2^N\setminus\{\varnothing\}$, and each $\mathbf{y}\in POSh(S,\mathbf{w}\mid_S,\succeq\mid_S)$, $\mathbf{y}_i \sim_i (\prod_{T\in 2^S\setminus\{\varnothing\}}(1+d_T))\mathbf{w}_i$ for all $i\in S$, which is equivalent to $((\pi\mid_S)\mathbf{y})_{\pi(i)} \sim^\pi_{\pi(i)} (\prod_{T\in 2^{\pi[S]}\setminus\{\varnothing\}}(1+d_T))\mathbf{w}_{\pi(i)}$ for all $i\in S$, i.e., $((\pi\mid_S)\mathbf{y})_j \sim^\pi_j (\prod_{T\in 2^{\pi[S]}\setminus\{\varnothing\}}(1+d_T))\mathbf{w}_j$ for all $j\in\pi[S]$. Hence, $(d'_{\pi[T]})_{T\in 2^N\setminus\{\varnothing\}}$ is a vector of dividend ratios for $\pi(N,\mathbf{w},\succeq)$ which, according to Theorem 1, implies that $\pi\mathbf{x}\in POSh\pi(N,\mathbf{w},\succeq)$.

To prove neutrality, again it is easy to see that the efficient allocation correspondence is neutral, that is, $\mathbf{x}\in E(N,\mathbf{w},\succeq)$ if and only if $\rho\mathbf{x}\in E\rho(N,\mathbf{w},\succeq)$, for all bijection $\rho : L \to L'$ and all $(N, \mathbf{w}, \succeq) \in \mathcal{E}$.

Consider the economies $(N, \mathbf{w}, \succeq)$ and $\rho(N, \mathbf{w}, \succeq)$, $\mathbf{x} \in POSh(N, \mathbf{w}, \succeq)$, and let $(d_T)_{T \in 2^N \setminus \{\varnothing\}}$ be its vector of dividend ratios. We show that $\rho\mathbf{x} \in POSh\rho(N, \mathbf{w}, \succeq)$ by proving that $(d'_T)_{T \in 2^N \setminus \{\varnothing\}}$, with $d'_T = d_T$ for all $T \in 2^N \setminus \{\varnothing\}$, constitutes a vector of dividend ratios for $\rho\mathbf{x}$. To see this, we note that for each $S \in 2^N \setminus \{\varnothing\}$ and each $\mathbf{y} \in POSh(S, \mathbf{w}\,|_S, \succeq|_S)$, $\mathbf{y}_i \sim_i (\prod_{T \in 2^S \setminus \{\varnothing\}} (1 + d_T))\mathbf{w}_i$ for all $i \in S$, which is equivalent to $(\rho\mathbf{y})_i \sim_i^{\rho} (\prod_{T \in 2^S \setminus \{\varnothing\}} (1 + d_T))(\rho\mathbf{w})_i$ for all $i \in S$. Therefore, $(d'_T)_{T \in 2^N \setminus \{\varnothing\}}$ is a vector of dividend ratios for $\rho(N, \mathbf{w}, \succeq)$; hence, $\rho\mathbf{x} \in POSh\rho(N, \mathbf{w}, \succeq)$ $\square$

Proof of Proposition 5. Consider two economies $(N, \mathbf{w}, \succeq), (N, \mathbf{w}', \succeq) \in \mathcal{E}$ such that $\mathbf{w}_i > \mathbf{w}'_i$ for $i \in N$ and $\mathbf{w}_j = \mathbf{w}'_j$ for each $j \in N \setminus \{i\}$. Using Theorem 1, we denote by $(d_S)_{S \in 2^N \setminus \{\varnothing\}}$ and $(d'_S)_{S \in 2^N \setminus \{\varnothing\}}$ their associated vectors of dividend ratios, respectively. We claim that $\prod_{\substack{T \ni i \\ T \subseteq S}} (1 + d_T) \geq \prod_{\substack{T \ni i \\ T \subseteq S}} (1 + d'_T)$ for all $S \subseteq N$ such that $S \ni i$. We prove the claim by induction on $|S|$. It trivially holds for $|S| = 1$.

For $S \subseteq N$ such that $|S| > 1$, suppose by contradiction that $\prod_{\substack{T \ni i \\ T \subseteq S}} (1 + d_T) < \prod_{\substack{T \ni i \\ T \subseteq S}} (1 + d'_T)$ and $\prod_{\substack{T \ni i \\ T \subseteq R}} (1 + d_T) \geq \prod_{\substack{T \ni i \\ T \subseteq R}} (1 + d'_T)$ for all $R \subsetneq S$. In particular, $\prod_{\substack{T \ni i \\ T \subseteq S \setminus \{j\}}} (1 + d_T) \geq \prod_{\substack{T \ni i \\ T \subseteq S \setminus \{j\}}} (1 + d'_T)$ for all $j \in S \setminus \{i\}$. Then, for each $j \in S \setminus \{i\}$, $\prod_{\substack{T \ni i,j \\ T \subseteq S}} (1 + d_T) < \prod_{\substack{T \ni i,j \\ T \subseteq S}} (1 + d'_T)$ since $\prod_{\substack{T \ni i \\ T \subseteq S}} (1 + d_T) = \prod_{\substack{T \ni i,j \\ T \subseteq S}} (1 + d_T) \prod_{\substack{T \ni i \\ T \subseteq S \setminus \{j\}}} (1 + d_T)$. Then, it is also true that $\prod_{\substack{T \ni j \\ T \subseteq S}} (1 + d'_T) = (\prod_{\substack{T \ni j \\ T \subseteq S \setminus \{i\}}} (1 + d'_T))(\prod_{\substack{T \ni i,j \\ T \subseteq S}} (1 + d'_T)) = (\prod_{\substack{T \ni j \\ T \subseteq S \setminus \{i\}}} (1 + d_T))(\prod_{\substack{T \ni i,j \\ T \subseteq S}} (1 + d'_T)) > \prod_{\substack{T \ni j \\ T \subseteq S}} (1 + d_T)$ for all $j \in S \setminus \{i\}$. Thus, $POSh_j(S, \mathbf{w}|_S, \succeq|_S) \prec_j POSh_j(S, \mathbf{w}'\,|_S, \succeq|_S)$ for all $j \in S$ (including i himself by premise), which is impossible. Therefore, the $POSh$ is not D-manipulable. $\square$

Proof of Proposition 6. Consider two economies $(N, \mathbf{w}, \succeq), (N, \mathbf{w}', \succeq) \in \mathcal{E}$ such that $\mathbf{w}_i > \mathbf{w}'_i$, $\mathbf{w}_i + \mathbf{w}_j = \mathbf{w}'_i + \mathbf{w}'_j$ for donor i and recipient j; $\mathbf{w}_k = \mathbf{w}'_k$ for each $k \in N \setminus \{i, j\}$. By Theorem 1, let $(d_T)_{T \in 2^N \setminus \{\varnothing\}}$ and $(d'_T)_{T \in 2^N \setminus \{\varnothing\}}$ be vectors of dividend ratios for economies $(N, \mathbf{w}, \succeq)$ and $(N, \mathbf{w}', \succeq)$, respectively. By considering the subeconomies without player j and without player i, Proposition 5 implies that $\prod_{\substack{T \ni i \\ T \subseteq N \setminus \{j\}}} (1 + d_T) \geq \prod_{\substack{T \ni i \\ T \subseteq N \setminus \{j\}}} (1 + d'_T)$ and $\prod_{\substack{T \ni j \\ T \subseteq N \setminus \{i\}}} (1 + d_T) \leq \prod_{\substack{T \ni j \\ T \subseteq N \setminus \{i\}}} (1 + d'_T)$.

Assume that the donor i is better off in $POSh(N, \mathbf{w}', \succeq)$ than in $POSh(N, \mathbf{w}, \succeq)$, which means that $\prod_{\substack{T \ni i \\ T \subseteq N}} (1 + d_T) < \prod_{\substack{T \ni i \\ T \subseteq N}} (1 + d'_T)$. The inequality implies that $\prod_{\substack{T \ni i,j \\ T \subseteq N}} (1 + d_T) < \prod_{\substack{T \ni i,j \\ T \subseteq N}} (1 + d'_T)$. Then $\prod_{\substack{T \ni j \\ T \subseteq N \setminus \{i\}}} (1 + d_T)(\prod_{\substack{T \ni i,j \\ T \subseteq N}} (1 + d_T)) <$

$(\prod_{\substack{T\ni j \\ T\subseteq N\setminus\{i\}}}(1+d'_T))(\prod_{\substack{T\ni i,j \\ T\subseteq N}}(1+d'_T))$, i.e., $\prod_{\substack{T\ni j \\ T\subseteq N}}(1+d_T) < \prod_{\substack{T\ni j \\ T\subseteq N}}(1+d'_T)$. Thus, recipient j must also be better off in $(N, \mathbf{w}', \succeq)$. Therefore, the transfer paradox is not possible for the $POSh$ $\square$

Proof of Proposition 7. To prove $\zeta = POSh$, we first show that $POSh \subseteq \zeta$, and then that ζ is essentially single-valued.

We prove that $POSh \subseteq \zeta$. Recall that there exists a vector of dividend ratios $(d_S)_{S\in 2^N\setminus\{\varnothing\}}$ such that $x \in POSh_i(N, \mathbf{w}, \succeq)$ if and only if $x \sim_i (\prod_{\substack{T\ni i \\ T\subseteq N}}(1+d_T))\mathbf{w}_i$ for all $i \in N$. Also, $POSh_j(N\setminus\{k\}, \mathbf{w}|_{N\setminus\{k\}}, \succeq|_{N\setminus\{k\}}) \sim_j (\prod_{\substack{T\ni j \\ T\subseteq N\setminus\{k\}}}(1+d_T))\mathbf{w}_j$ for all $k \in N$ and all $j \in N\setminus\{k\}$. Take $a_{ij} \equiv \prod_{\substack{T\ni j \\ T\subseteq N\setminus\{i\}}}(1+d_T)$ and $c_{ij} \equiv \prod_{\substack{T\ni i,j \\ T\subseteq N}}(1+d_T)$ for all $j \in N\setminus\{i\}$ and all $i \in N$. Then, x together with the vectors $(c_{ij})_{j\in N\setminus\{i\}}$ and $(a_{ij})_{j\in N\setminus\{i\}}$ for all $i \in N$, satisfy part (2b) of Definition 13. Moreover, $c_{ij} = c_{ji}$ for all $j \in N\setminus\{i\}$ and all $i \in N$. Hence, part (2a) of Definition 13 also holds. Therefore, $POSh \subseteq \zeta$.

We prove that ζ is essentially single-valued by induction on the number of agents $|N|$. It trivially holds for $|N| = 1$ by definition. For $|N| > 1$, we hypothesize that ζ is essentially single-valued for any economy with $n-1$ agents. It implies that a^N_{ij} in Definition 13 is unique (that is, it is the same for all $\mathbf{x} \in \zeta(N, \mathbf{w}, \succeq)$) for all $i, j \in N$ such that $i \neq j$.

Consider any $\mathbf{x} \in \zeta(N, \mathbf{w}, \succeq)$. According to (2b) in Definition 13, it is the case that $\mathbf{x}_j \sim_j c^N_{ij}a^N_{ij}\mathbf{w}_j$ and $\mathbf{x}_j \sim_j c^N_{kj}a^N_{kj}\mathbf{w}_j$, for all $j \in N$ and all $i, k \in N\setminus\{j\}$ such that $i \neq k$. Then, strong monotonicity implies that $c^N_{ij}a^N_{ij} = c^N_{kj}a^N_{kj}$ for all $i, k \in N\setminus\{j\}$ such that $i \neq k$.

Therefore, we have $|N|(|N|-1)$ equations: $c^N_{i1}n = \frac{a^N_{n1}}{a^N_{i1}}c^N_{n1}$ for all $i \in N\setminus\{1\}$, and $c^N_{ij} = \frac{a^N_{(j-1)j}}{a^N_{ij}}c^N_{(j-1)j}$ for all $i \in N\setminus\{j\}$ and all $j \in N\setminus\{1\}$. By substituting them in condition $(2a)$, $\prod_{j\in N\setminus\{i\}} c^N_{ij} = \prod_{j\in N\setminus\{i\}} c^N_{ji}$, we have $\frac{a^N_{n1}}{a^N_{i1}}c^N_{n1}[\prod_{j\in N\setminus\{1,i\}}\frac{a^N_{(j-1)j}}{a^N_{ij}}c^N_{(j-1)j}] = \prod_{j\in N\setminus\{i\}}\frac{a^N_{(i-1)i}}{a^N_{ji}}c^N_{(i-1)i}$ for $i \in N\setminus\{1\}$. From this equality we obtain that $c^N_{(i-1)i} = \frac{1}{a^N_{(i-1)i}}\sqrt[n-1]{\frac{a^N_{n1}}{a^N_{i1}}c^N_{n1}[\prod_{j\in N\setminus\{1,i\}}\frac{a^N_{(j-1)j}}{a^N_{ij}}c^N_{(j-1)j}][\prod_{j\in N\setminus\{i\}}a^N_{ji}]}$ for each $i \in N\setminus\{1\}$. It means that $c^N_{(i-1)i}$ can be expressed as an increasing function of $(c^N_{(k-1)k})_{k\in N\setminus\{1,i\}}$ and c^N_{n1} as its arguments, for each $i \in N\setminus\{1\}$. Moreover, by repeated substitution, we can represent each $c^N_{(i-1)i}$ as an increasing function of c^N_{n1} solely, for all $i = N\setminus\{1\}$. Hence,

$c^N_{(i-1)i}$ takes the form $c^N_{(i-1)i}(c^N_{n1})$ for $i = 2, \ldots, n$. Thus, for all $\mathbf{x} \in \zeta(N, \mathbf{w}, \succeq)$, $\mathbf{x}_1 \sim_1 c^N_{n1} a^N_{n1} \mathbf{w}_1$, and $\mathbf{x}_i \sim_i c^N_{(i-1)i}(c^N_{n1}) a^N_{(i-1)i} \mathbf{w}_i$ for all $i = 2, \ldots, n$. By Pareto efficiency of $\mathbf{x}$, c_{n1} is unique. Therefore, ζ is essentially single-valued, which completes the proof. $\square$

Proof of Theorem 2. To formalize our argument, we introduce the following notation. Applying the mechanism to an economy $(N, \mathbf{w}, \succeq)$ results in an extensive form game, which is denoted by $\mathit{\Gamma}(N, \mathbf{w}, \succeq)$. Denote the SPNE outcome correspondence by $\mathcal{SN}$, which enables us to express the set of all SPNE outcomes of an extensive form game as the value of $\mathcal{SN}$ at this game. For example, the set of all SPNE outcomes of $\mathit{\Gamma}(N, \mathbf{w}, \succeq)$ is $\mathcal{SN}\mathit{\Gamma}(N, \mathbf{w}, \succeq)$. Furthermore, if $\mathbf{x} \sim \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{SN}\mathit{\Gamma}(N, \mathbf{w}, \succeq)$, we may write $\mathcal{SN}_i\mathit{\Gamma}(N, \mathbf{w}, \succeq)$ and compare it with a bundle in terms of $\succeq_i$ for all $i \in N$ without incurring confusion. We may also consider the subgames of $\mathit{\Gamma}(N, \mathbf{w}, \succeq)$. We denote by $\mathit{\Gamma}^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq)$ the subgame starting from the information set after the proposer i's allocation plan is rejected and the bids made were $\mathbf{b}^N$. In particular, the bids made for i were b^N_{ji} for all $j \in N$.

The proof has three parts: (i) for all $(N, \mathbf{w}, \succeq) \in \mathcal{E}^H$, all $\mathbf{x} \in \mathcal{SN}\mathit{\Gamma}(N, \mathbf{w}, \succeq)$, and all $\mathbf{y} \in E(N, \mathbf{w}, \succeq)$, then $\mathbf{y} \in \mathcal{SN}\mathit{\Gamma}(N, \mathbf{w}, \succeq)$ if $\mathbf{y} \sim \mathbf{x}$; (ii) $\mathcal{SN}\mathit{\Gamma}(N, \mathbf{w}, \succeq) \subseteq POSh(N, \mathbf{w}, \succeq)$ for every $(N, \mathbf{w}, \succeq) \in \mathcal{E}^H$; and (iii) $\mathcal{SN}\mathit{\Gamma}(N, \mathbf{w}, \succeq) \neq \varnothing$ for every $(N, \mathbf{w}, \succeq) \in \mathcal{E}^H$. Note that parts (i)-(iii) imply that $\mathcal{SN}\mathit{\Gamma}(N, \mathbf{w}, \succeq) = POSh(N, \mathbf{w}, \succeq)$.

We prove the three parts simultaneously by induction on $|N|$. The case where $|N| = 1$ is trivial, so we restrict attention to the cases where $|N| \geq 2$. We assume the induction hypothesis that (i)-(iii), and consequently the implementation of $POSh$ by the bidding mechanism, hold for all economies with less than n agents.

To prove part (i), we first state and prove two claims. We notice that the set of agents in $\mathit{\Gamma}^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq)$ is N where the set of agents in $\mathit{\Gamma}(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ is $N \setminus \{i\}$. However, the sets of SPNE of the extensive-form games $\mathit{\Gamma}^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq)$ and $\mathit{\Gamma}(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ are "similar" in the following sense:

Claim 1. *Given* $\mathbf{b}^N$ *and* $\mathbf{y}' \in Z(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$, *define* $\mathbf{y} \in Z(N, \mathbf{w}, \succeq)$

by

$$\mathbf{y}_j = \begin{cases} \left(\frac{\sqrt[n]{B_i^N}}{b_{ji}^N}\right)\mathbf{y}'_j & \text{if } j \in N\setminus\{i\}; \\ \mathbf{w}_i - \sum_{k\in N\setminus\{i\}}\left(\frac{\sqrt[n]{B_i^N}}{b_{ki}^N} - 1\right)\mathbf{y}'_k & \text{if } j = i. \end{cases}$$

Then $\mathbf{y} \in \mathcal{SN}\Gamma^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq)$ *if* $\mathbf{y}' \in \mathcal{SN}\Gamma(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$.

We prove Claim 1. Let $(s_k)_{k\in N\setminus\{i\}}$ be an SPNE strategy profile for $\Gamma(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ whose outcome is $\mathbf{y}'$. The final outcome in $\Gamma^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq)$ if the agents play $(s_k)_{k\in N\setminus\{i\}}$ is $\mathbf{y}$ (note that although the game $\Gamma^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq)$ involves all the agents in N, only those in $N\setminus\{i\}$ choose a strategy) . We prove that $(s_k)_{k\in N\setminus\{i\}}$ is an SPNE for $\Gamma^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq)$. Suppose otherwise. Let s'_j be a profitable deviation for $j \in N\setminus\{i\}$ in $\Gamma^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq)$. After the deviation, j obtains a bundle $\mathbf{z}_j$ which is multiplied by $\left(\frac{\sqrt[n]{B_i^N}}{b_{ji}^N}\right)$ and such that $\left(\frac{\sqrt[n]{B_i^N}}{b_{ji}^N}\right)\mathbf{z}_j \succ_j \mathbf{y}_j = \left(\frac{\sqrt[n]{B_i^N}}{b_{ji}^N}\right)\mathbf{y}'_j$. Since the preferences are homothetic, we have that $\mathbf{z}_j \succ_j \mathbf{y'}_j$. However, if j deviates in $\Gamma(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ from $(s_k)_{k\in N\setminus\{i\}}$ by choosing s'_j, then he obtains the allocation $\mathbf{z}_j$, which he prefers to $\mathbf{y}'_j$. This is not possible because $\mathbf{y}' \in \mathcal{SN}\Gamma(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$. Hence, $(s_k)_{k\in N\setminus\{i\}}$ is an SPNE for $\Gamma^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq)$ and $\mathbf{y} \in \mathcal{SN}\Gamma^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq)$, which concludes the proof of Claim 1.

We note that following the same arguments as in the proof of Claim 1, the reverse result also holds. That is, given $\mathbf{b}^N$ and $\mathbf{y} \in \mathcal{SN}\Gamma^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq)$, define $\mathbf{y}'$ by $\mathbf{y}'_j = \left(\frac{b_{ji}^N}{\sqrt[n]{B_i^N}}\right)\mathbf{y}_j$ for all $j \in N\setminus\{i\}$. We note that $\mathbf{y}' \in Z(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ because $\mathbf{y}'_j$ is the allocation that player j obtains in the game $\Gamma^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq)$ before the rejected proposer i transfers the bundles according to $\mathbf{b}^N$ (see Case II at $t = 3$ as described in the proportional bidding mechanism); hence, $\sum_{k\in N\setminus\{i\}}\mathbf{y}'_k = \sum_{k\in N\setminus\{i\}}\mathbf{w}_k$. Then $\mathbf{y}' \in \mathcal{SN}\Gamma(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ if $\mathbf{y} \in \mathcal{SN}\Gamma^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq)$.

By the induction hypothesis, we have that $\mathcal{SN}\Gamma(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) = POSh(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$. Then, we use Claim 1 and its reverse to obtain that $\mathcal{SN}_j\Gamma^{-i}_{\mathbf{b}^N}(N, \mathbf{w}, \succeq) = \frac{\sqrt[n]{B_i^N}}{b_{ji}^N}POSh_j(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ for all $j \in N\setminus\{i\}$. With the aid of this equality, we prove that every SPNE outcome can be supported by an SPNE that leads to an immediate agreement in Claim 2.

Claim 2. *For every SPNE outcome* $\mathbf{x} \in \mathcal{SN}\Gamma(N, \mathbf{w}, \succeq)$, *take an SPNE whose outcome is* $\mathbf{x}$. *Let* $\mathbf{b}^N$ *be the profile of the agents' bid vectors in that SPNE and* *consider the subgame where agent* $i \in \operatorname{argmax}_{k \in N} B_k^N$ *becomes the proposer. Then,*

a) $\mathbf{x}_j \sim_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ *for all* $j \in N \setminus \{i\}$.

b) There exists an SPNE where:

b1) each agent $j \in N \setminus \{i\}$ *accepts any i's allocation plan* $\mathbf{z} \in \mathbb{R}_+^{(N\setminus\{i\})\times L}$ *if* $\mathbf{z}_j \succeq_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ *and rejects it otherwise;*

b2) the proposer i puts forth an efficient allocation plan $\mathbf{z}$ *such that* $\mathbf{z}_j \sim_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ *for each agent* $j \in N \setminus \{i\}$.

We prove part a) of Claim 2. Notice that, for any economy $(N, \mathbf{w}, \succeq) \in \mathcal{E}^H$, given a profile of bid vectors $\mathbf{b}^N$ and a proposer $i \in N$, then agent $j \in N \setminus \{i\}$ accepts at equilibrium any i's allocation plan $\mathbf{z} \in \mathbb{R}_+^{(N\setminus\{i\})\times L}$ if $\mathbf{z}_j \succ_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ and rejects it if $\mathbf{z}_j \prec_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$. This holds because agent j obtains a bundle $\frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N\setminus\{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ in case of rejection (by the induction hypothesis and Claim 1). Moreover, a proposal $\mathbf{z}$ such that $\mathbf{z}_j \succ_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ cannot be part of an SPNE. Such a proposal would be accepted, but i could lower $\mathbf{z}_j$ by a sufficiently small amount and propose another acceptable offer resulting in a higher residual bundle for himself. Therefore, at equilibrium, j necessarily obtains a bundle that makes him indifferent to the bundle $\frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$. Hence, the equation in part a) holds.

The previous arguments also prove that the strategy proposed in part b1) of the claim can be part of an equilibrium. To prove part b2), consider an efficient allocation plan $\mathbf{z}$ such that $\mathbf{z}_j \sim_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) \sim_j \mathbf{x}_j$ for each agent $j \in N \setminus \{i\}$. Part b1) ensures that the agents in $N \setminus \{i\}$ will accept this proposal. Moreover, given that it is efficient, there is no better allocation for i that would be accepted. Proposing a rejected plan cannot be a profitable deviation for agent i because rejection leads to a feasible allocation where every $j \in N \setminus \{i\}$ obtains a bundle equivalent for him to $\frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N\setminus\{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$, which cannot be strictly better for i than $\mathbf{z}$. Hence, Claim 2 is proven.

We now prove part (i) of our induction. Take an allocation $\mathbf{y} \in E(N, \mathbf{w}, \succeq)$ such that $\mathbf{y} \sim \mathbf{x}$. Then consider the strategy profile that is identical to that in Claim 2 (including the bids) except that agent i proposes $\mathbf{y} \mid_{N\setminus\{i\}}$ in b2). Given that the SPNE is an ordinal solution, the strategy profile described in Claim 2 is an SPNE if and only if the new strategy profile is an SPNE. Therefore, for all SPNE outcome $\mathbf{x} \in \mathcal{SN}\Gamma(N, \mathbf{w}, \succeq)$ and all $\mathbf{y} \in E(N, \mathbf{w}, \succeq)$ such that $\mathbf{y} \sim \mathbf{x}$, $\mathbf{y} \in \mathcal{SN}\Gamma(N, \mathbf{w}, \succeq)$, which completes part (i).

We proceed to prove part (ii). We first establish the following property of the equilibrium bids.

Claim 3. *In any SPNE, $B_i^N = 1$ for all $i \in N$. Moreover, each agent $i \in N$ is indifferent about the identity of the proposer.*

We first show that each $i \in N$ is indifferent about the identity of the proposer among the agents in $\operatorname{argmax}_{j\in N} B_j^N$. Let $|\operatorname{argmax}_{j\in N} B_j^N| = p$. If $p = 1$, then this assertion automatically holds. If $p \geq 2$, then assume that agent i strictly prefers k to z as the proposer, for a pair of agents $k, z \in \operatorname{argmax}_{j\in N} B_j^N$ (where i could possibly be either k or z). In this case, agent i has an incentive to deviate by increasing b_{ik}^N to $(1+\epsilon)b_{ik}^N$ and decreasing b_{iz}^N to $\frac{b_{iz}^N}{1+\epsilon}$, where $\epsilon \in \mathbb{R}_{++}$ is sufficiently small. This deviation would ensure that agent k would become the proposer. There are two cases. Case (a): If $i \neq k$ then agent i avoids the positive probability of receiving a bundle strictly worse than $\frac{\sqrt[n]{B_k^N}}{b_{ik}^N} POSh_i(N \setminus \{k\}, \mathbf{w} \mid_{N\setminus\{k\}}, \succeq\mid_{N\setminus\{k\}})$ and ensures receiving a bundle equivalent for him to $\frac{\sqrt[n]{B_k^N}}{(1+\epsilon)b_{ik}^N} POSh_i(N \setminus \{k\}, \mathbf{w} \mid_{N\setminus\{k\}}, \succeq\mid_{N\setminus\{k\}})$, by Claim 2. This is a profitable deviation if ϵ is small enough. Case (b): If $i = k$ then agent i becomes the proposer. He can put forth an allocation plan such that each agent $j \in N \setminus \{i\}$ is assigned a bundle $\sim_j$-equivalent to $\frac{\sqrt[n]{(1+\epsilon)B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w} \mid_{N\setminus\{i\}}, \succeq\mid_{N\setminus\{i\}})$, and the plan will be accepted, by Claim 2. As before, by continuity of $\succeq_i$, agent i is strictly better off by switching to the new bid vector for a sufficiently small ϵ. In either case, it is profitable for agent i to switch to the new bid vector. Therefore, every agent must be indifferent concerning the identity of the proposer.

Second, suppose that $B_i^N = 1$ does not hold for all $i \in N$, which implies that there is $m \in N \setminus \operatorname{argmax}_{i\in N} B_i^N$. Let $p = |\operatorname{argmax}_{i\in N} B_i^N|$. Then, any agent $j \in \operatorname{argmax}_{i\in N} B_i^N$ could switch to the following new bid vector $\tilde{\mathbf{b}}_j^N$, which is well-defined:

$$\tilde{b}^N_{jk} \equiv \begin{cases} (1-\epsilon)b^N_{jk} & \text{if } k = j, \\ (1-\epsilon)^2 b^N_{jk} & \text{if } k \in \operatorname{argmax}_{i\in N} B^N_i \setminus \{j\}, \\ (1-\epsilon)^{1-2p} b^N_{jk} & \text{if } k = m, \\ b^N_{jk} & \text{otherwise,} \end{cases}$$

where $\epsilon \in \mathbb{R}_{++}$ is sufficiently small. After this switch, agent j would be the proposer for sure. Notice that $\frac{\sqrt[n]{\tilde{B}^N_j}}{b^N_{ij}} < \frac{\sqrt[n]{B^N_j}}{b^N_{ij}}$ for each $i \in N \setminus \{j\}$ because $\tilde{B}^N_j < B^N_j$ given that $\tilde{b}^N_{jj} < b^N_{jj}$. Then, by Claim 2, agent j can propose an allocation plan that assigns a bundle slightly better for i than $\frac{\sqrt[n]{\tilde{B}^N_j}}{b^N_{ij}} POSh_i(N \setminus \{j\}, \mathbf{w}\,|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}})$ instead of $\frac{\sqrt[n]{B^N_j}}{b^N_{ij}} POSh_i(N \setminus \{j\}, \mathbf{w}\,|_{N\setminus\{j\}}, \succeq|_{N\setminus\{j\}})$, for each agent $i \in N \setminus \{j\}$, and the plan will be accepted. Given that $\frac{\sqrt[n]{\tilde{B}^N_j}}{b^N_{ij}} < \frac{\sqrt[n]{B^N_j}}{b^N_{ij}}$, agent $j \in N \setminus \{i\}$ is strictly better off after the switch, which is not possible. Thus, $\operatorname{argmax}_{i\in N} B^N_i = N$, i.e., $B^N_i = B^N_j$ for all $i, j \in N$. Since $\prod_{i\in N} B^N_i = \prod_{i\in N}\prod_{j\in N} b^N_{ji} = \prod_{j\in N}\prod_{i\in N} b^N_{ji} = 1^n = 1$, then $B^N_i = 1$ for all $i \in N$. This concludes the proof of Claim 3.

To continue with the proof of part (ii), let

$$c^N_{ij} \equiv \frac{\sqrt[n]{B^N_i}}{b^N_{ji}}, \tag{4}$$

for all $i, j \in N$ such that $i \neq j$. We verify that $\frac{\sqrt[n]{B^N_i}}{b_{ii}} \prod_{j\in N\setminus\{i\}} c^N_{ij} = \frac{B^N_i}{\prod_{j\in N} b^N_{ji}} = 1$ and $\frac{\sqrt[n]{B^N_i}}{b_{ii}} \prod_{j\in N\setminus\{i\}} c^N_{ji} = \prod_{k\in N} \frac{\sqrt[n]{B^N_k}}{b_{ik}} = \frac{\sqrt[n]{\prod_{k\in N} B^N_k}}{\prod_{k\in N} b_{ik}} = \sqrt[n]{\prod_{k\in N} B^N_k} = \sqrt[n]{\prod_{k\in N}\prod_{j\in N} b^N_{jk}} = \sqrt[n]{\prod_{j\in N}\prod_{k\in N} b^N_{jk}} = 1$. Thus, $\prod_{j\in N\setminus\{i\}} c^N_{ij} = \prod_{j\in N\setminus\{i\}} c^N_{ji} = \frac{b_{ii}}{\sqrt[n]{B^N_i}}$, which satisfies the condition $(2a)$ of Definition 13 of the $POSh$. To check that the concessions that we defined in (4) also satisfy the condition $(2b)$ of Definition 13, we notice that, when agents' preferences are homothetic, this condition is equivalent to:

For each $j \in N \setminus \{i\}$, $\mathbf{x}_j \sim_j c^N_{ij}\zeta_j(N \setminus \{i\}, \mathbf{w}|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$.

We can interpret $\zeta_j(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ as $POSh_j(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}})$ since, as we have shown, $\mathcal{SN}_j\Gamma(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}, \succeq|_{N\setminus\{i\}}) = POSh_j(N\setminus\{i\}, \mathbf{w}\,|_{N\setminus\{i\}}$ $, \succeq|_{N\setminus\{i\}})$. Then, condition $(2b)$ is also satisfied, which completes the proof of part (ii).

We now prove part (iii) of our induction. Let us denote by $\mathbf{c}^{N'}_i$, for all $i \in N'$, the unique vector of concessions for the subeconomy $(N', \mathbf{w}\,|_{N'}, \succeq|_{N'})$ for all $N' \subseteq N$

such that $n' \geq 2$. We construct the agents' strategy profile as follows. At any subgame where the remaining set of active agents (i.e., the agents who choose a strategy) is $N' \subseteq N$ and they have to bid, agent $i \in N'$ selects the bid $b_{ij}^{N'} = \frac{1}{c_{ji}^{N'}}$ for player $j \in N' \setminus \{i\}$ and $b_{ii}^{N'} = \prod_{k \in N' \setminus \{i\}} c_{ki}^{N'}$ (hence, the bids are well-defined because $B_i^{N'} = \prod_{j \in N'} b_{ji}^{N'} = \prod_{j \in N' \setminus \{i\}} \frac{1}{c_{ij}^{N'}} \prod_{k \in N' \setminus \{i\}} c_{ki}^{N'} = \prod_{j \in N' \setminus \{i\}} \frac{1}{c_{ji}^{N'}} \prod_{k \in N' \setminus \{i\}} c_{ki}^{N'} = 1$ due to the condition 2a) of Definition 13). Proposers' equilibrium allocation plans and the rest of agents' responses to the proposers' plans at any subgame follow the description in Claim 2 b).

We have shown that no agent has an incentive to deviate once the bids $\mathbf{b}^N$ have been made. It remains to verify that no agent has an incentive to change the bid vector. Suppose that agent $i \in N$ changes his bid from $\mathbf{b}_i^N$ to $\tilde{\mathbf{b}}_i^N$. Then it will not be the case that $B_j^N = 1$ for all $j \in N$. Denote by α the resulting proposer. Given that $\tilde{B}_\alpha^N \equiv \tilde{b}_{i\alpha}^N \prod_{j \in N \setminus \{i\}} b_{j\alpha}^N > \prod_{j \in N} b_{j\alpha}^N = B_\alpha^N$, it is necessarily the case that $\tilde{b}_{i\alpha}^N > b_{i\alpha}^N$. If $\alpha = i$, then each agent $j \in N \setminus \{i\}$ will be allocated a bundle $\mathbf{x}_j \sim_j \frac{\sqrt[n]{\tilde{B}_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w} \mid_{N \setminus \{i\}}, \succeq\mid_{N \setminus \{i\}}) \succeq_j \frac{\sqrt[n]{B_i^N}}{b_{ji}^N} POSh_j(N \setminus \{i\}, \mathbf{w} \mid_{N \setminus \{i\}}, \succeq\mid_{N \setminus \{i\}})$ (see Claim 2 a)), and he will be better off. By Pareto efficiency of the final allocation, agent i, as the residual claimant, cannot be strictly better off. If, on the other hand, $\alpha \neq i$, agent i will be allocated a bundle $\mathbf{x}_i \sim_i \frac{\sqrt[n]{\tilde{B}_\alpha^N}}{\tilde{b}_{i\alpha}^N} POSh_i(N \setminus \{\alpha\}, \mathbf{w} \mid_{N \setminus \{\alpha\}}, \succeq\mid_{N \setminus \{\alpha\}}) \preceq_i \frac{\sqrt[n]{B_\alpha^N}}{b_{i\alpha}^N} POSh_i(N \setminus \{\alpha\}, \mathbf{w} \mid_{N \setminus \{\alpha\}}, \succeq\mid_{N \setminus \{\alpha\}})$ because $\tilde{b}_{i\alpha}^N > b_{i\alpha}^N$ implies that $\frac{\sqrt[n]{\tilde{B}_\alpha^N}}{\tilde{b}_{i\alpha}^N} < \frac{\sqrt[n]{B_\alpha^N}}{b_{i\alpha}^N}$. Therefore, agent i cannot be strictly better off either. This proves the existence of an SPNE, which concludes the proof of the theorem.

□

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